DRONACHARYA
College of Engineering

Digital Signal Processing

Unit –IV
Implementation of Discrete Time Systems
Multirate Digital Signal Processing:

Content

❖ **Implementation of Discrete Time Systems:**

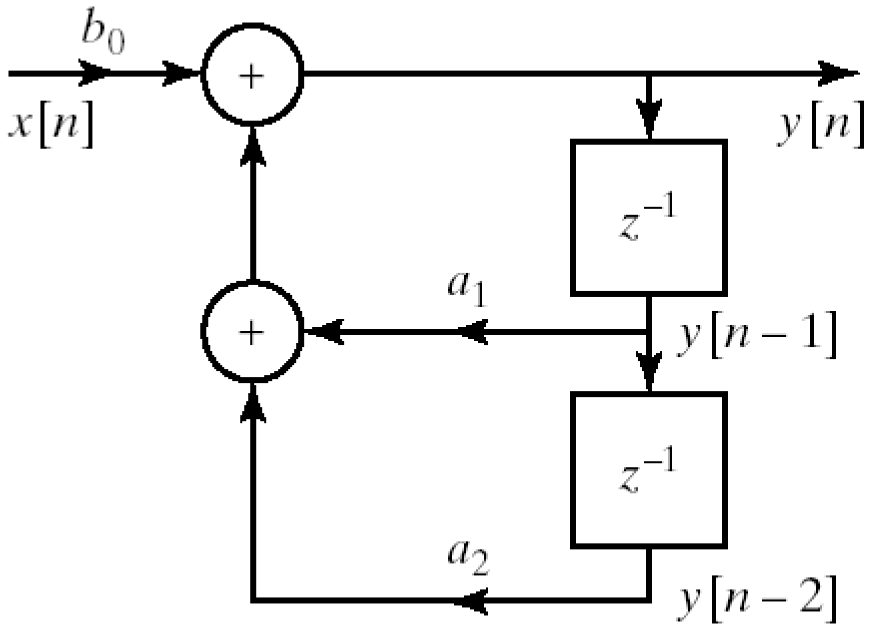
- Block diagrams and signal flow graphs for FIR and IIR systems
- Direct form, Cascade form
- Frequency Sampling Structures and Lattice structures for FIR systems
- Direct form, Cascade form, Parallel form, and Lattice and Lattice-Ladder Structures for IIR systems
- Representation of fixed point and floating point numbers,
- Finite word length effects,
- Parametric and non-parametric spectral estimation.
- Applications of Digital Signal Processing

❖Multirate Digital Signal Processing

- Introduction to multirate digital signal processing
- Multi rate structures for sampling rate conversion
- Multistage decimator and interpolator
- Polyphase decomposition
- Digital Filter Banks.

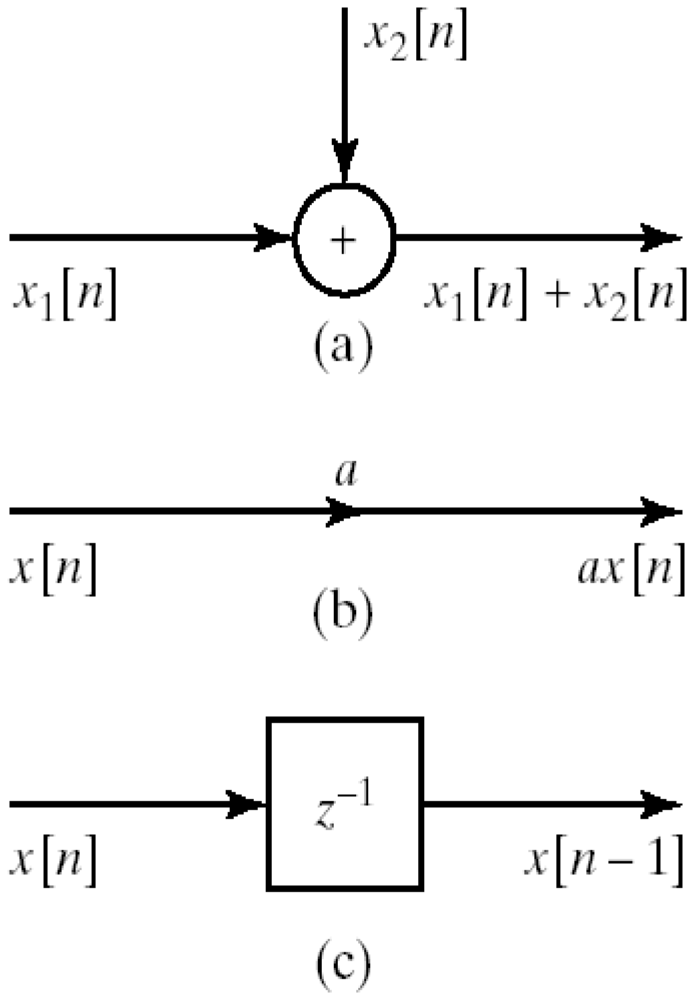
Example

- Block diagram representation of $y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]$



Block Diagram Representation

- LTI systems with rational system function can be represented as constant-coefficient difference equation
- The implementation of difference equations requires delayed values of the
  - input
  - output
  - intermediate results
- The requirement of delayed elements implies need for storage
- We also need means of
  - addition
  - multiplication



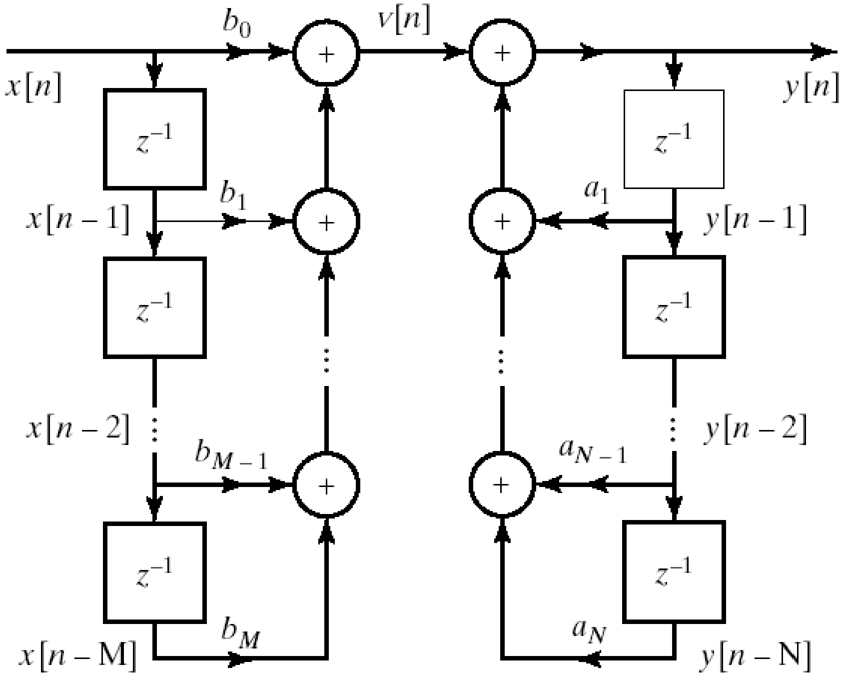
Direct Form I

$$\sum_{k=0}^{N} \hat{a}_k y[n-k] = \sum_{k=0}^{M} \hat{b}_k x[n-k]$$

- General form of difference equation

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Alternative equival



Direct Form I

- Transfer function can be written as

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = H_2(z)H_1(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$V(z) = H_1(z)X(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)X(z)$$

$$v[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Direct Form I Represents

$$Y(z) = H_2(z)V(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right)V(z)$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n]$$

Alternative Representation

- Replace order of cascade LTI systems

$$H(z) = H_1(z)H_2(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right)$$

$$W(z) = H_2(z)X(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}}\right)X(z)$$

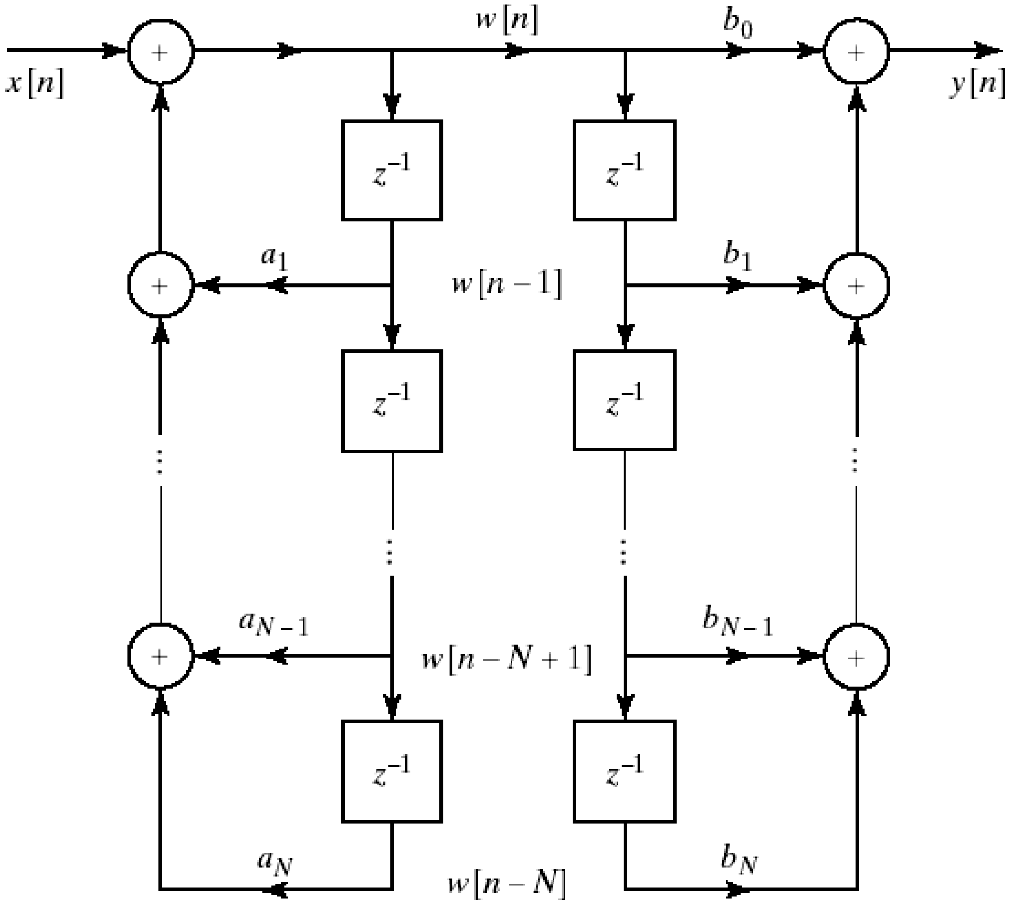
$$Y(z) = H_1(z)W(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)W(z)$$

$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

Alternative Block Diagram

- We can change the order o

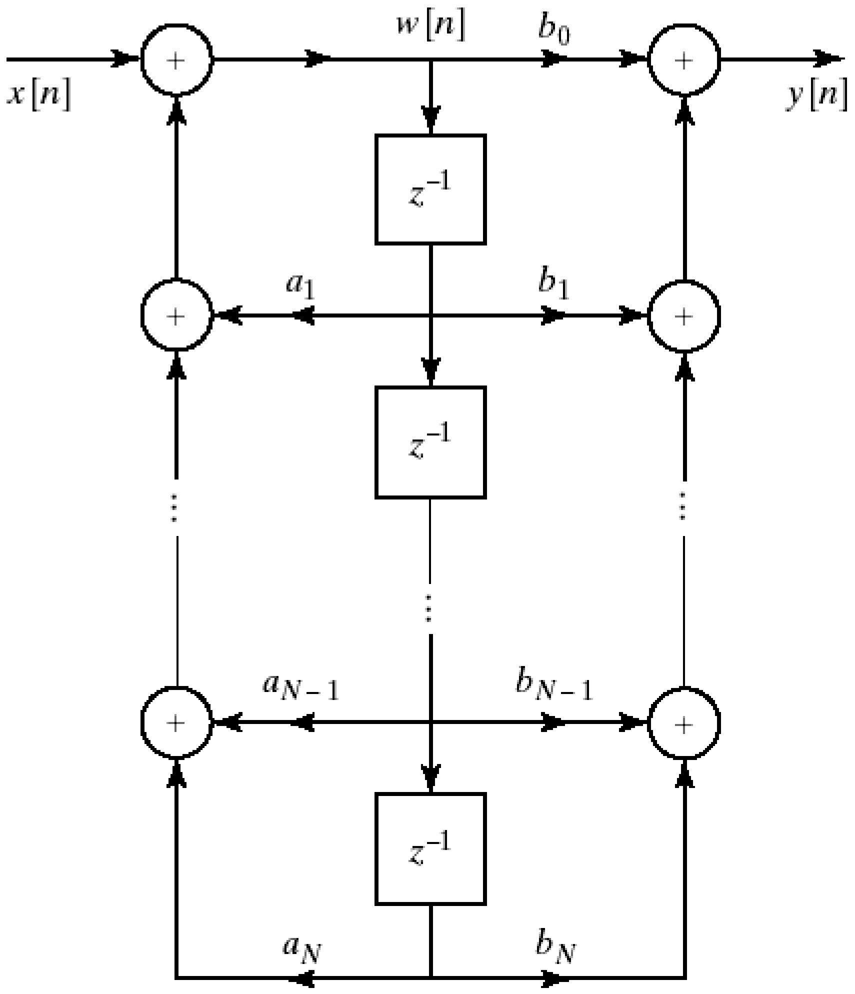


$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

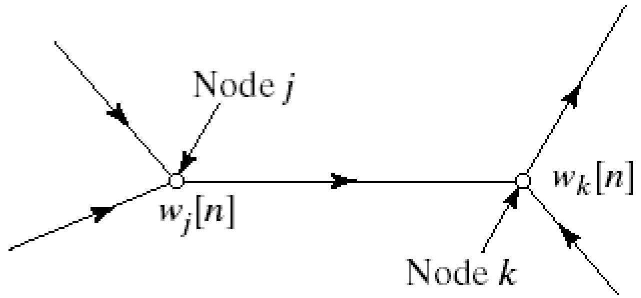
Direct Form II

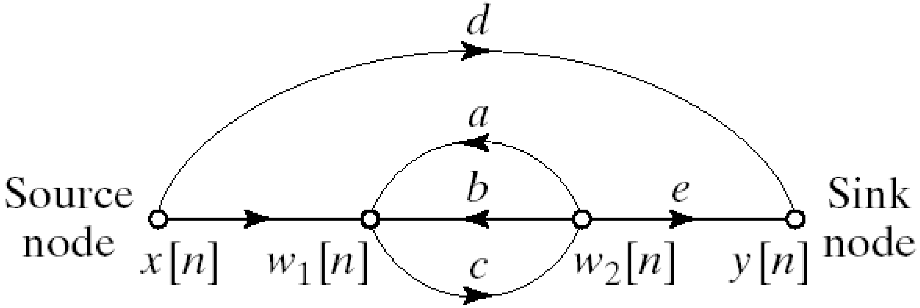
- No need to store the same data twice in previous system
- So we can collapse the delay elements into one chain
- This is called Direct Form II or the Canonical Form
- Theoretically no difference between Direct Form I and II
- Implementation wise
  - Less memory in Direct II
  - Difference when using finite-precision arithmetic



Signal Flow Graph Representation

- Similar to block diagram representation
  - Notational differences
- A network of directed branches connected at n

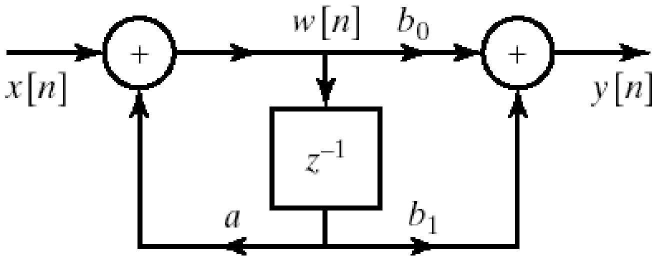




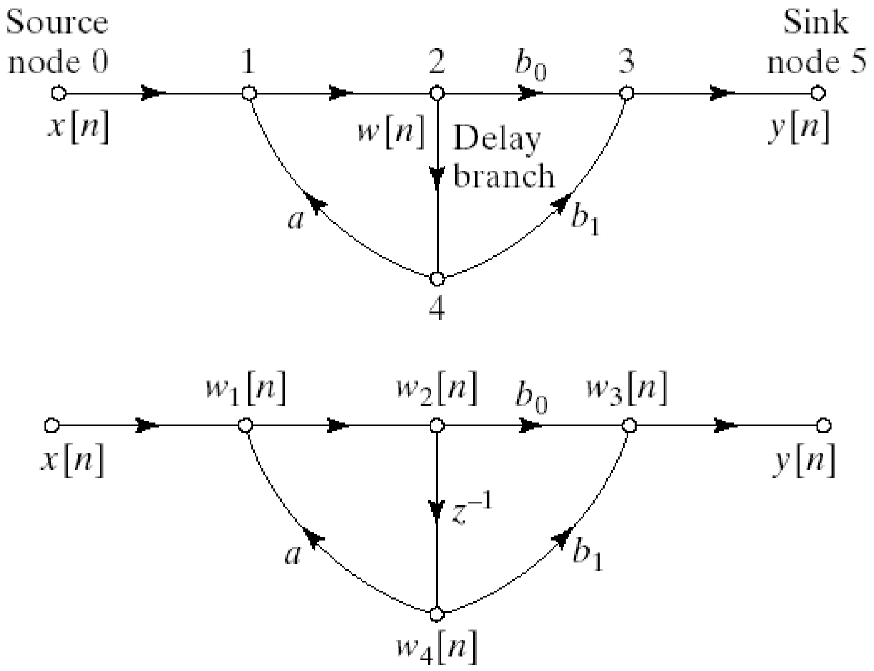
- Example representation of a difference equation

- Representation of Direct Form II with signal flow graphs

Example



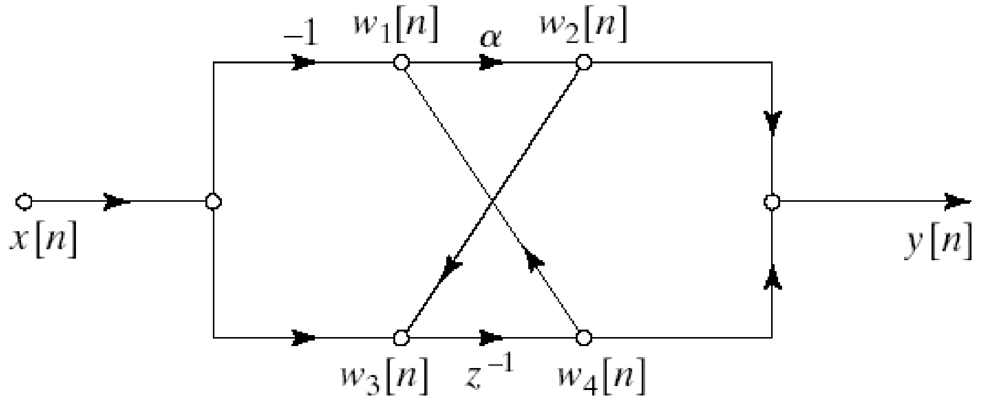
(a)



$$w_1[n] = aw_4[n] + x[n]$$
$$w_2[n] = w_1[n]$$
$$w_3[n] = b_0 w_2[n] + b_1 w_4[n]$$
$$w_4[n] = w_2[n-1]$$
$$y[n] = w_3[n]$$

$$w_1[n] = aw_1[n-1] + x[n]$$
$$y[n] = b_0 w_1[n] + b_1 w_1[n-1]$$

Determination of System Function from Flow Graph



$$w_1[n] = w_4[n] - x[n]$$
$$w_2[n] = \alpha w_1[n]$$
$$w_3[n] = w_2[n] + x[n]$$
$$w_4[n] = w_3[n-1]$$
$$y[n] = w_2[n] + w_4[n]$$

$$W_1(z) = W_4(z) - X(z)$$
$$W_2(z) = \alpha W_1(z)$$
$$W_3(z) = W_2(z) + X(z)$$
$$W_4(z) = W_3(z) z^{-1}$$
$$Y(z) = W_2(z) + W_4(z)$$

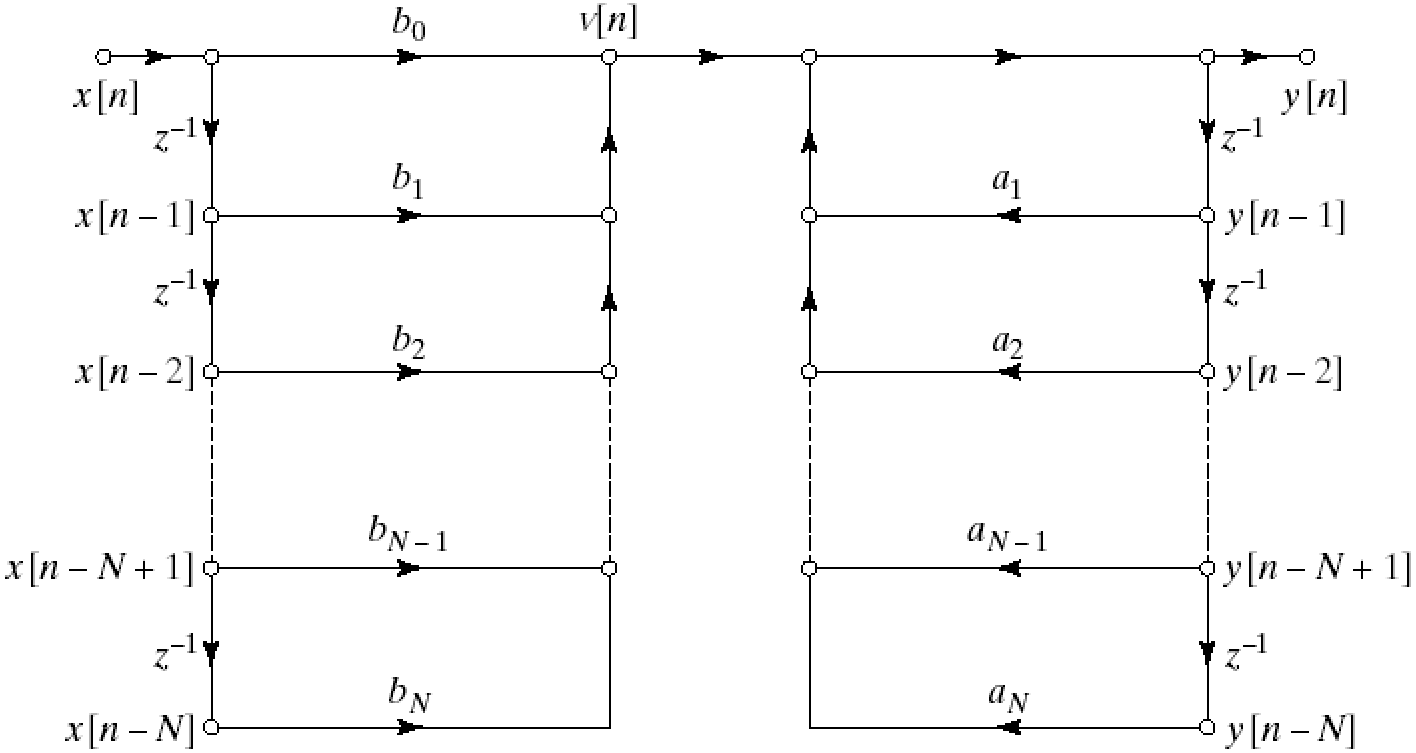
$\longrightarrow$

$$W_2(z) = \frac{\alpha X(z)(z^{-1} - 1)}{1 - \alpha z^{-1}}$$
$$W_4(z) = \frac{X(z) z^{-1}(1 - \alpha)}{1 - \alpha z^{-1}}$$
$$Y(z) = W_2(z) + W_4(z)$$

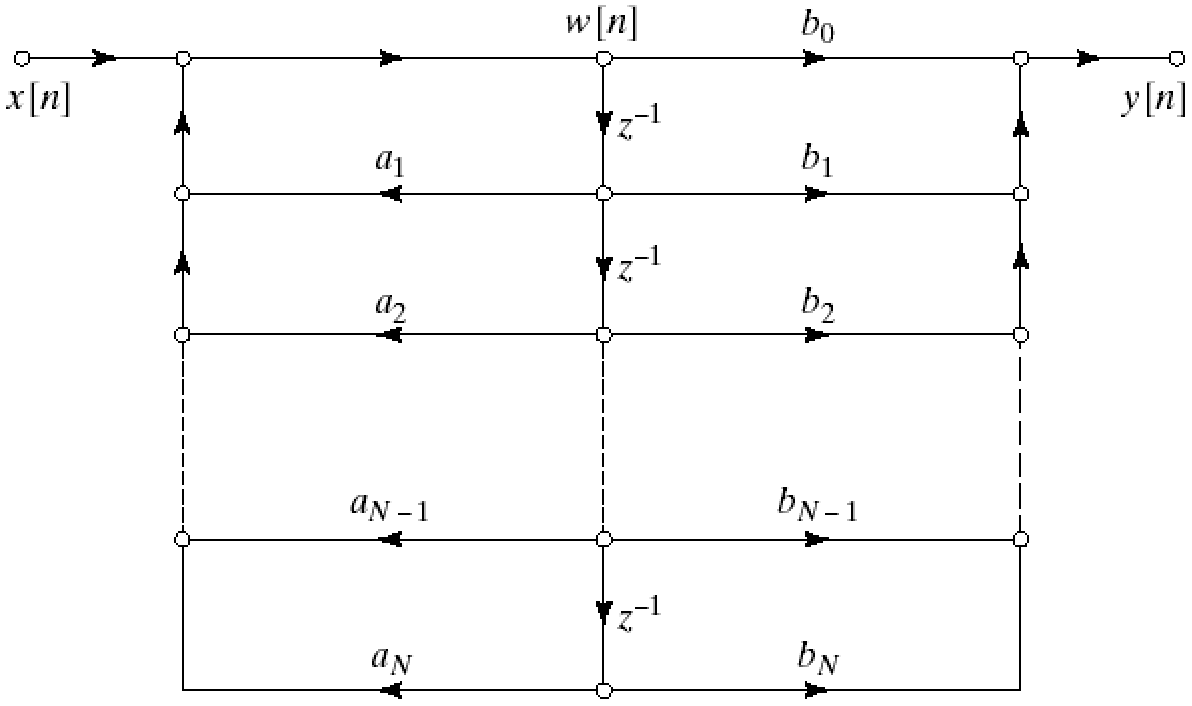
$\longrightarrow$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$
$$h[n] = \alpha^{n-1} u[n-1] - \alpha^{n+1} u[n]$$

Basic Structures for IIR Systems: Direct Form I



Basic Structures for IIR Systems: Direct Form II



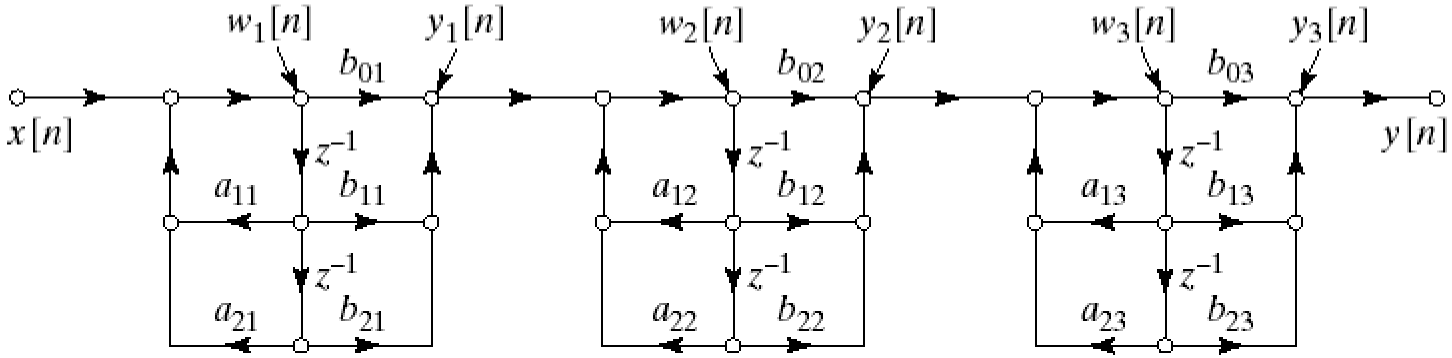
Basic Structures for IIR Systems: Cascade Form

$$H(z) = A \frac{\prod_{k=1}^{M_1}\left(1 - f_k z^{-1}\right)\prod_{k=1}^{M_2}\left(1 - g_k z^{-1}\right)\left(1 - g_k^* z^{-1}\right)}{\prod_{k=1}^{N_1}\left(1 - c_k z^{-1}\right)\prod_{k=1}^{N_2}\left(1 - d_k z^{-1}\right)\left(1 - d_k^* z^{-1}\right)}$$

- General form for cascade implementation

$$H(z) = \prod_{k=1}^{M_1} \frac{b_{0k} + b_{1k} z^{-1} - b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

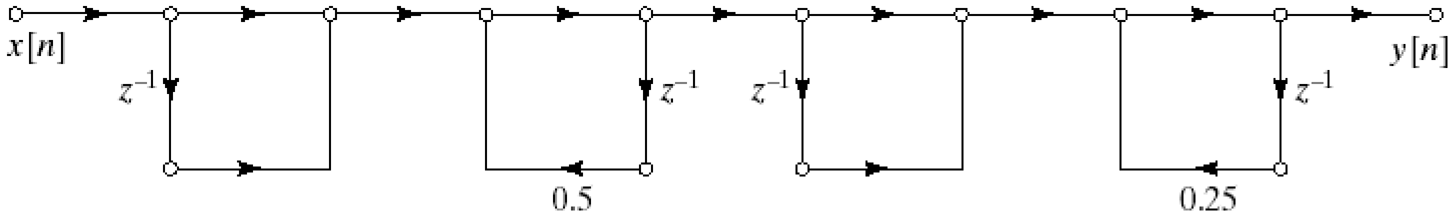
- More practical form in 2nd order systems



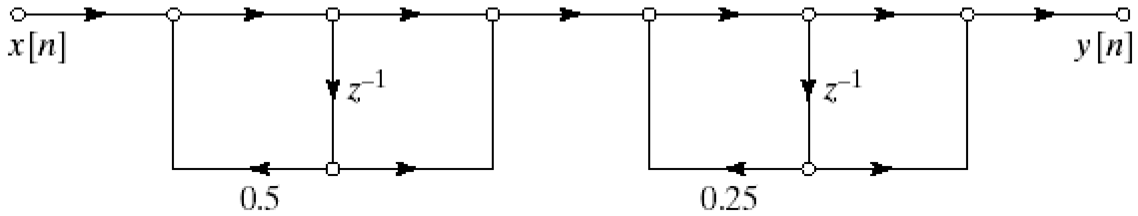
Example

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$

$$= \frac{(1 + z^{-1})}{(1 - 0.5z^{-1})} \frac{(1 + z^{-1})}{(1 - 0.25z^{-1})}$$



- Cascade of Direct Form I subsections



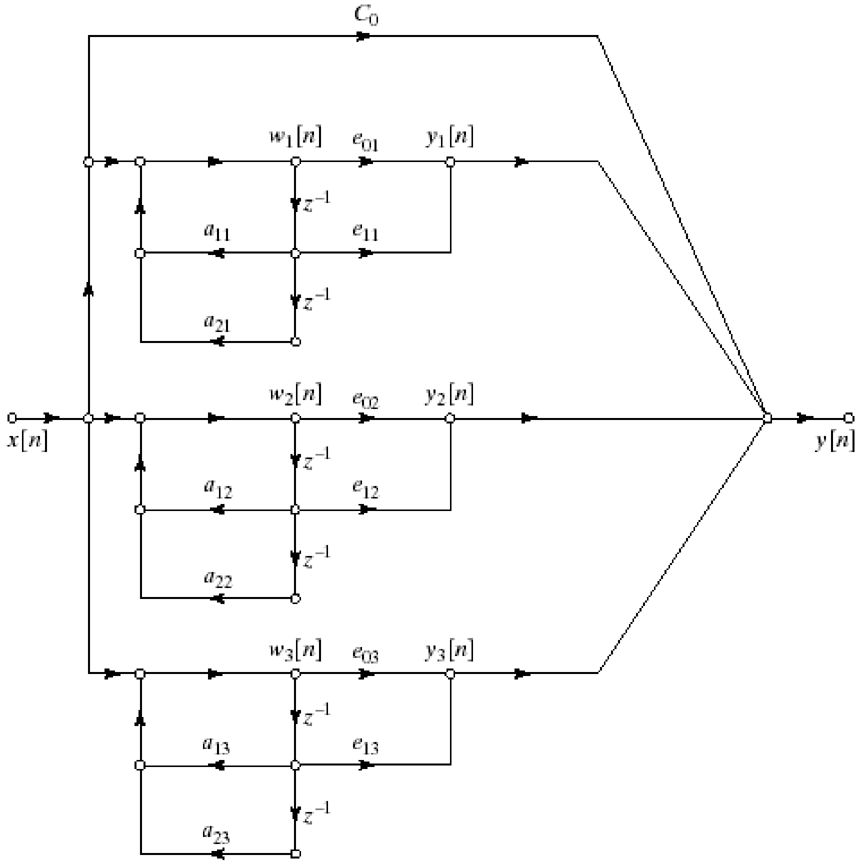
- Cascade of Direct Form II subsections

Basic Structures for IIR Systems: Parallel Form

$$H(z) = \sum_{k=0}^{N_P} C_k z^{-k} + \sum_{k=1}^{N_P} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_P} \frac{B_k\left(1 - e_k z^{-1}\right)}{\left(1 - d_k z^{-1}\right)\left(1 - d_k^* z^{-1}\right)}$$

- Represent system function using partial fraction expansion

$$H(z) = \sum_{k=0}^{N_P} C_k z^{-k} + \sum_{k=1}^{N_S} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$



- Or by pairing the real poles

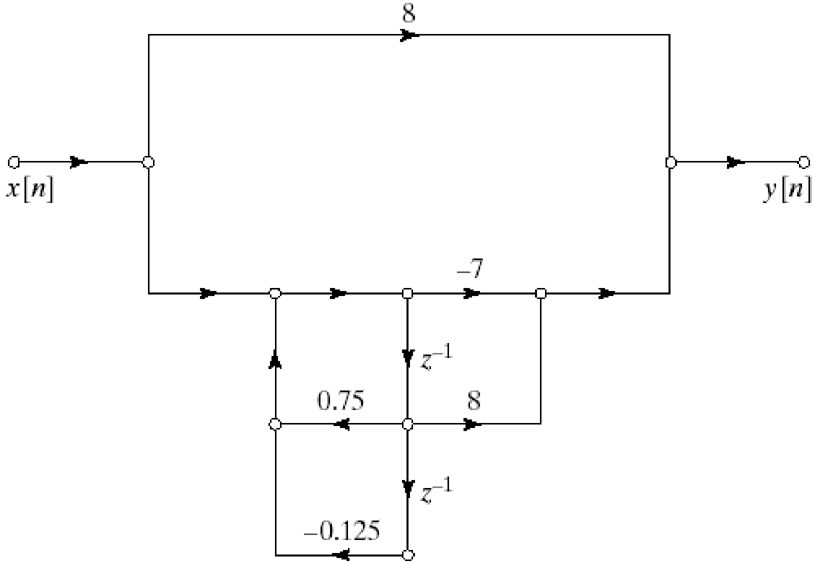
Example

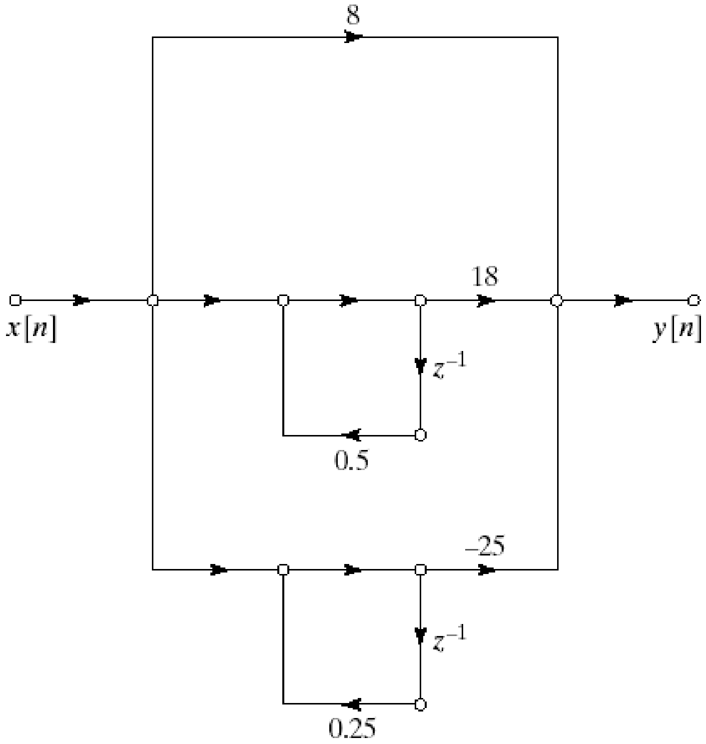
$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = 8 + \frac{18}{(1 - 0.5z^{-1})} - \frac{25}{(1 - 0.25z^{-1})}$$

- Partial Fraction Expansion

$$H(z) = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

- Combine

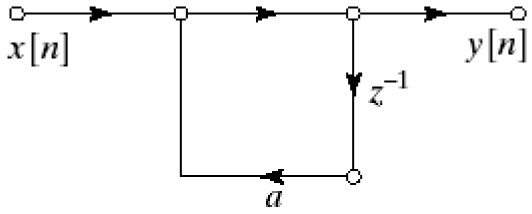




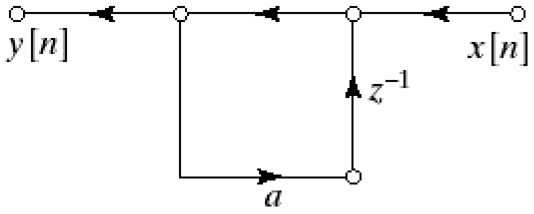
Transposed Forms

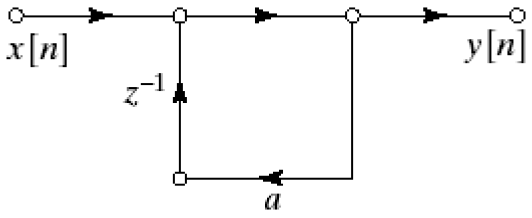
- Linear signal flow graph property:
  - Transposing doesn't change the input-output relation
- Transposing:
  - Reverse directions of all branches
  - Interchange input and output nodes

$$H(z) = \frac{1}{1 - az^{-1}}$$



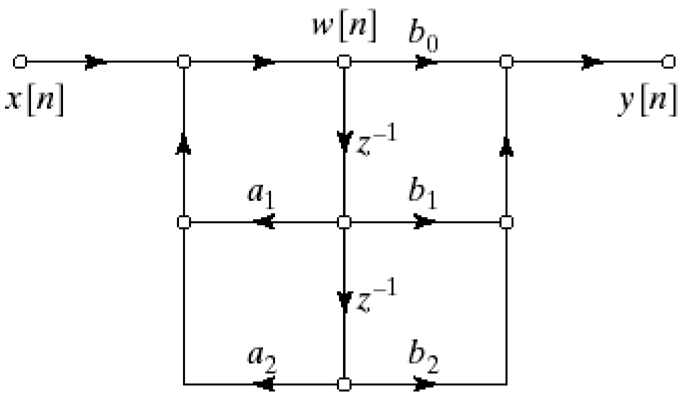
- Example:



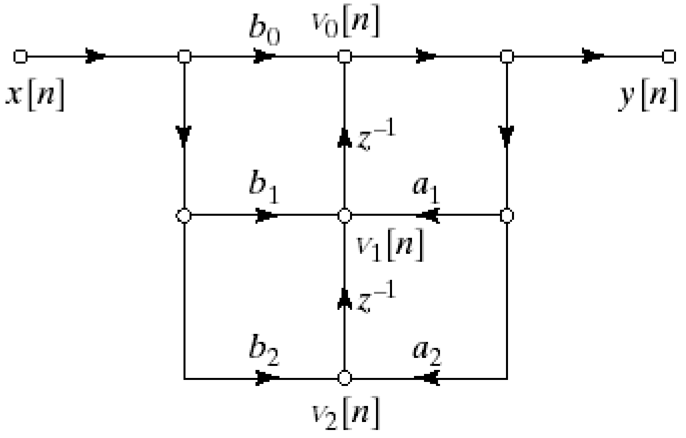


  - Reverse directions of branches and interchange input and output

Example



Transpose →

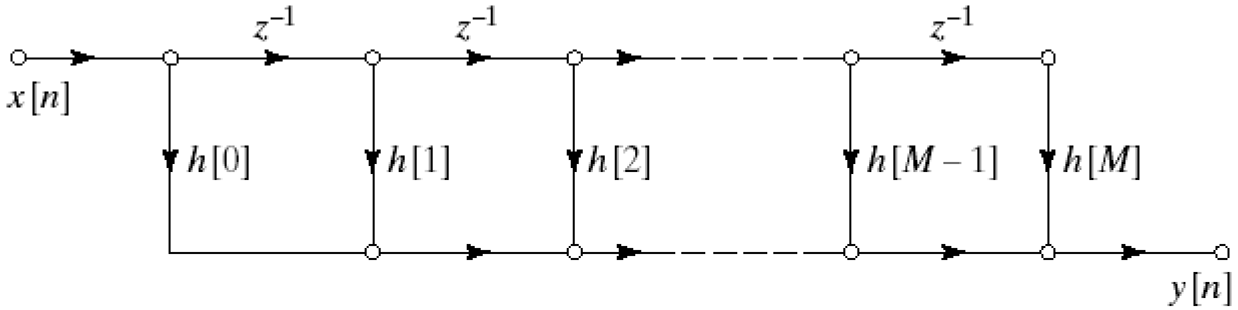


$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

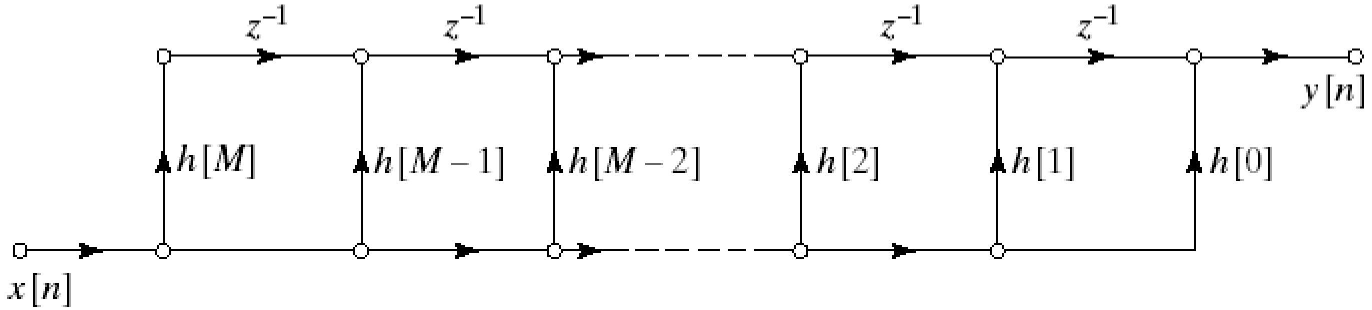
- Both have the same system function or difference equation

Basic Structures for FIR Systems: Direct Form

- Special cases of IIR direct form



- Transpose of direct form I gives direct form II
- Both forms are equal for FIR systems

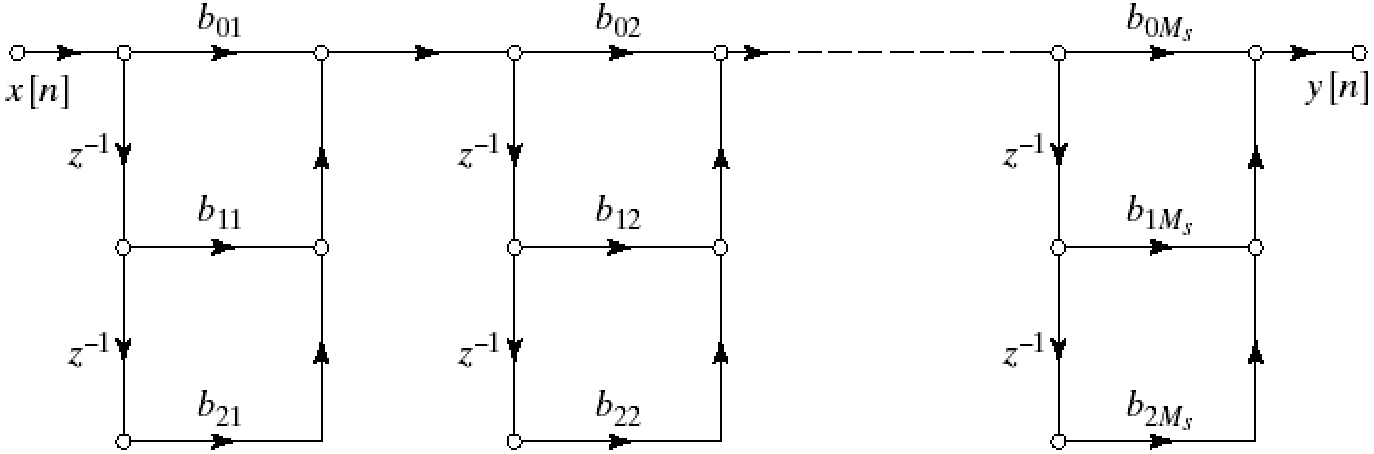


- Tapped delay line

Basic Structures for FIR Systems: Cascade Form

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = \prod_{k=1}^{M_S} \left(b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2} \right)$$

- Obtained by factoring the polynomial system function



Structures for Linear-Phase FIR Systems

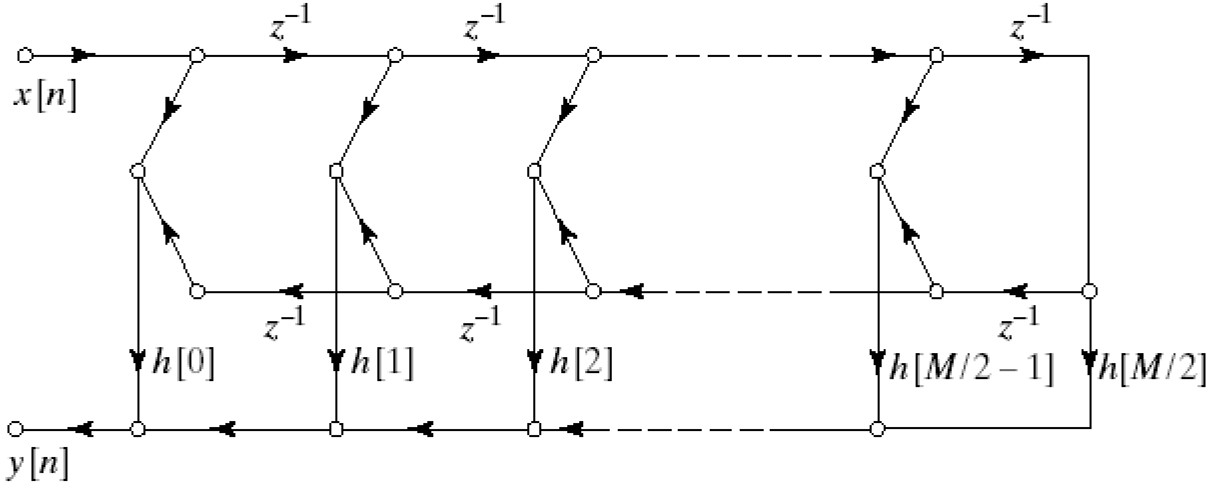
$$h[M-n] = h[n] \qquad n = 0,1,\ldots,M \quad \text{(type I or III)}$$
$$h[M-n] = -h[n] \qquad n = 0,1,\ldots,M \quad \text{(type II or IV)}$$

- Causal FIR system with generalized linear phase are symmetric:
- Symmetry means we can half the number of multiplications
- Example: For even M and type I or type III systems:

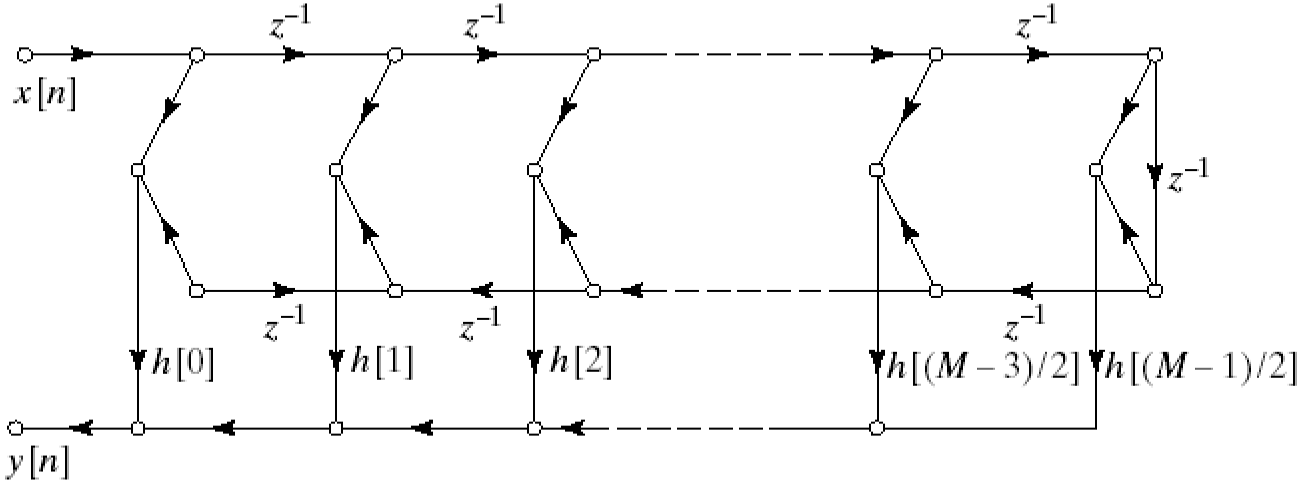
$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=M/2+1}^{M} h[k]x[n-k]$$
$$= \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=0}^{M/2-1} h[M-k]x[n-M+k]$$
$$= \sum_{k=0}^{M/2-1} h[k]\left(x[n-k] + x[n-M+k]\right) + h[M/2]x[n-M/2]$$

Structures for Linear-Phase FIR Systems

- Structure for even M



- Structure for odd M



Attraction of DSP comes from key advantages such as :

* Guaranteed accuracy: (accuracy is only determined by the number of bits used)
* Perfect Reproducibility: Identical performance from unit to unit
  ie. A digital recording can be copied or reproduced several times with no loss in signal quality
* No drift in performance with temperature and age
* Uses advances in semiconductor technology to achieve:
  (i) smaller size
  (ii) lower cost
  (iii) low power consumption
  (iv) higher operating speed
* Greater flexibility: Reprogrammable , no need to modify the hardware
* Superior performance
  ie. linear phase response can be achieved
  complex adaptive filtering becomes possible

Disadvantages of DSP

* Speed and Cost

  DSP techniques are limited to signals with relatively low bandwidths

  DSP designs can be expensive, especially when large bandwidth signals are involved.

  ADC or DACs are either to expensive or do not have sufficient resolution for wide bandwidth applications.

* DSP designs can be time consuming plus need the necessary resources (software etc)

* Finite word-length problems

  If only a limited number of bits is used due to economic considerations serious degradation in system performance may result.

- The use of finite precision arithmetic makes it necessary to quantize filter calculations by rounding or truncation.
- Roundoff noise is that error in the filter output that results from rounding or truncating calculations within the filter.
- As the name implies, this error looks like low-level noise at the filter output

Application Areas

Image Processing	Instrumentation/Control	Speech/Audio	Military
Pattern recognition	spectrum analysis	speech recognition	secure communications
Robotic vision	noise reduction	speech synthesis	radar processing
Image enhancement	data compression	text to speech	sonar processing
Facsimile	position and rate control	digital audio	missile guidance
animation		equalization	

Telecommunications	Biomedical	Consumer applications
Echo cancellation	patient monitoring	cellular mobile phones
Adaptive equalization	scanners	UMTS
ADPCM trans-coders	EEG brain mappers	digital television
Spread spectrum	ECG Analysis	digital cameras
Video conferencing	X-Ray storage/enhancement	internet phone
		etc.

Multirate Digital Signal Processing

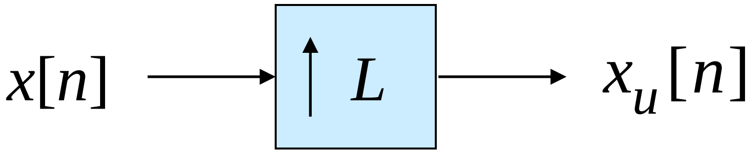
Basic Sampling Rate Alteration Devices

- ***Up-sampler*** - Used to increase the sampling rate by an integer factor
- ***Down-sampler*** - Used to decrease the sampling rate by an integer factor

Up-Sampler

Time-Domain Characterization

- An up-sampler with an ***up-sampling factor*** *L*, where *L* is a positive integer, develops an output sequence $x_u[n]$ with a sampling rate that is *L* times larger than that of the input sequence $x[n]$
- Block-diagram representation



Up-Sampler

- Up-sampling operation is implemented by inserting $L-1$ equidistant zero-valued samples between two consecutive samples of $x[n]$
- Input-output relation $x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$

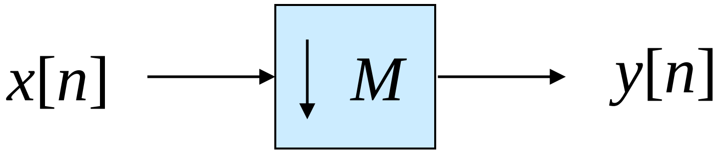
Up-Sampler

- In practice, the zero-valued samples inserted by the up-sampler are replaced with appropriate nonzero values using some type of filtering process
- Process is called *interpolation* and will be discussed later

Down-Sampler

Time-Domain Characterization

- An down-sampler with a ***down-sampling factor*** M, where M is a positive integer, develops an output sequence $y[n]$ with a sampling rate that is $(1/M)$-th of that of the input sequence $x[n]$
- Block-diagram representation



Down-Sampler

- Down-sampling operation is implemented by keeping every M-th sample of $x[n]$ and removing $M-1$ in-between samples to generate $y[n]$
- Input-output relation

$$y[n] = x[nM]$$

Basic Sampling Rate Alteration Devices

- Sampling periods have not been explicitly shown in the block-diagram representations of the up-sampler and the down-sampler
- This is for simplicity and the fact that the ***mathematical theory of multirate systems*** can be understood without bringing the sampling period T or the sampling frequency F_T into the picture

Down-Sampler

- Figure below shows explicitly the time-dimensions for the down-sampler

$$x[n] = x_a(nT) \longrightarrow \boxed{\downarrow M} \longrightarrow y[n] = x_a(nMT)$$

Input sampling frequency

$$F_T = \frac{1}{T}$$

Output sampling frequency

$$F_T' = \frac{F_T}{M} = \frac{1}{T'}$$

Up-Sampler

- Figure below shows explicitly the time-dimensions for the up-sampler

$$x[n] = x_a(nT) \longrightarrow \boxed{\uparrow L} \longrightarrow y[n]$$

$$= \begin{cases} x_a(nT/L), & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Input sampling frequency

$$F_T = \frac{1}{T}$$

Output sampling frequency

$$F_T' = LF_T = \frac{1}{T'}$$

Basic Sampling Rate Alteration Devices

- The ***up-sampler*** and the ***down-sampler*** are *linear* but *time-varying discrete-time systems*
- We illustrate the time-varying property of a down-sampler
- The time-varying property of an up-sampler can be proved in a similar manner

Basic Sampling Rate Alteration Devices

- Consider a factor-of-*M* down-sampler defined by
- Its output $y[n] = x[n M]$ for an input $x_1[n] = x[n - n_0]$ is then $y_1[n]$ given by $y_1[n] = x_1[Mn] = x[Mn - n_0]$
- From the input-output relation of the down-sampler we obtain

$$y[n - n_0] = x[M(n - n_0)]$$
$$= x[Mn - Mn_0] \neq y_1[n]$$

Up-Sampler

Frequency-Domain Characterization

- Consider first a factor-of-2 up-sampler whose input-output relation in the time-domain is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Up-Sampler

- In terms of the z-transform, the input-output relation is then given by

$$X_u(z) = \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \ \ z^{-2m} = X(z^2)$$

Up-Sampler

- In a similar manner, we can show that for a ***factor-of-L up-sampler*** $X_u(z) = X(z^L)$
- On the unit circle, for $z = e^{j\omega}$, the input-output relation is given by

$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

Down-Sampler

Frequency-Domain Characterization

- Applying the *z*-transform to the input-output relation of a factor-of-*M* down-sampler

$$y[n] = x[Mn]$$

we get

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n}$$

- The expression on the right-hand side cannot be directly expressed in terms of $X(z)$

Down-Sampler

- To get around this problem, define a new sequence $x_{\text{int}}[n]$:

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- Then

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn] z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_{\text{int}}[k] z^{-k/M} = X_{\text{int}}(z^{1/M})$$

Down-Sampler

- Now, $x_{\text{int}}[n]$ can be formally related to $x[n]$ through $x_{\text{int}}[n] = c[n] \cdot x[n]$

  where $c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \ldots \\ 0, & \text{otherwise} \end{cases}$

- A convenient representation of $c[n]$ is given by

$$c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn}$$

  where

$$W_M = e^{-j2\pi/M}$$

Down-Sampler

- Taking the *z*-transform of $x_{\text{int}}[n] = c[n] \cdot x[n]$ and making use of

$$c[n] = \frac{1}{M}\sum_{k=0}^{M-1} W_M^{kn}$$

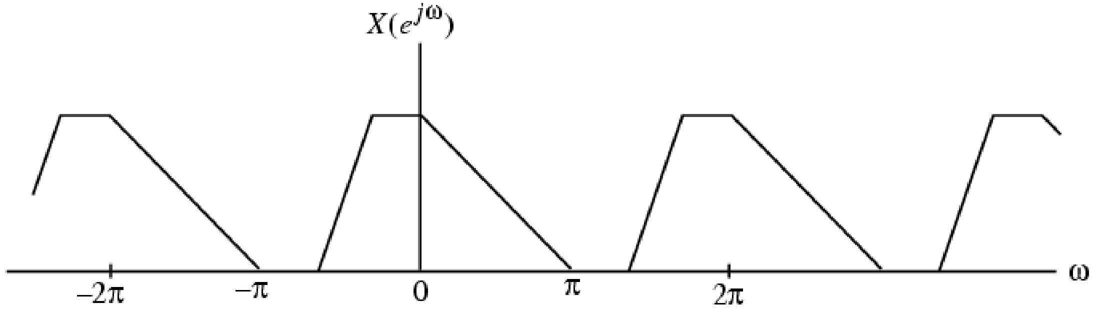
we arrive at

$$X_{\text{int}}(z) = \sum_{n=-\infty}^{\infty} c[n]x[n]z^{-n} = \frac{1}{M}\sum_{n=-\infty}^{\infty}\left(\sum_{k=0}^{M-1} W_M^{kn}\right)x[n]z^{-n}$$

$$= \frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{n=-\infty}^{\infty} x[n]W_M^{kn}z^{-n}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW_M^{-k}\right)$$

Down-Sampler

- Consider a factor-of-2 down-sampler with an input $x[n]$ whose spectrum is as shown below

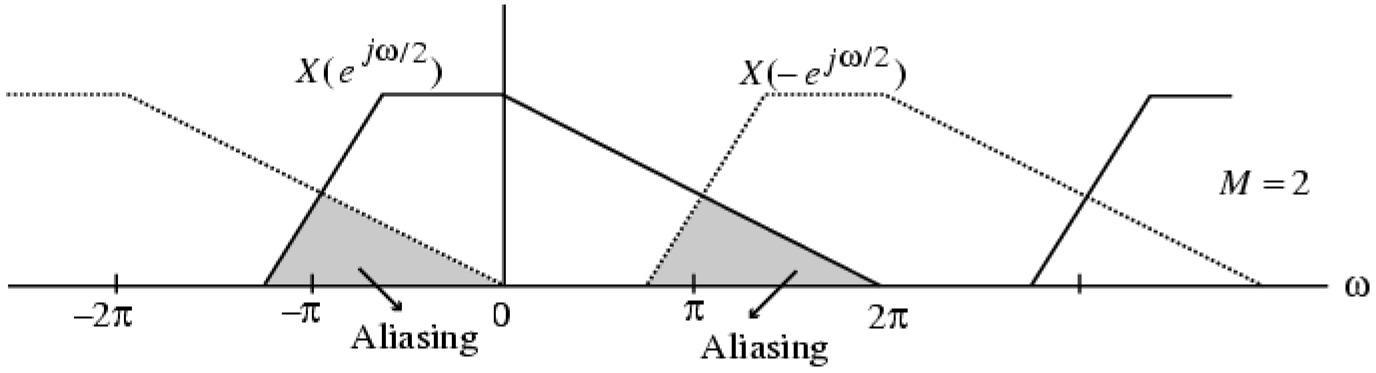


- The DTFTs of the output and the input sequences of this down-sampler are then related as

$$Y(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\}$$

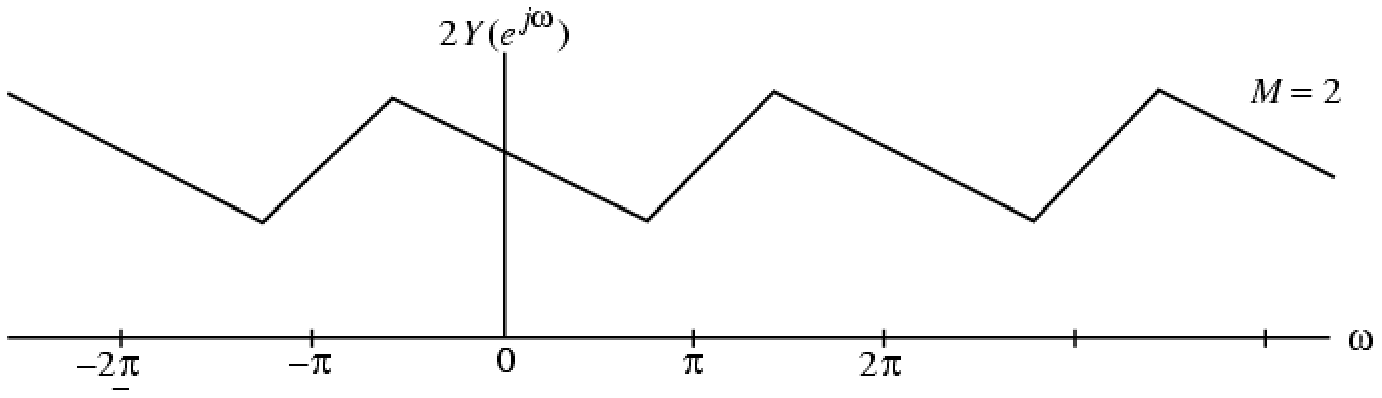
Down-Sampler

- Now $X(-e^{j\omega/2}) = X(e^{j(\omega-2\pi)/2})$ implying that the second term $X(-e^{j\omega/2})$ in the previous equation is simply obtained by shifting the first term $X(e^{j\omega/2})$ to the right by an amount 2π as shown below



Down-Sampler

- The plots of the two terms have an overlap, and hence, in general, the original "*shape*" of $X(e^{j\omega})$ is lost when $x[n]$ is down-sampled as indicated below



Down-Sampler

- This overlap causes the ***aliasing*** that takes place due to under-sampling
- There is no overlap, i.e., no aliasing, only if

$$X(e^{j\omega}) = 0 \qquad \text{for } |\omega| \geq \pi/2$$

- Note: $X(e^{j\omega})$ is indeed periodic with a period 2π, even though the stretched version of $Y(e^{j\omega})$ is periodic with a period 4π

Down-Sampler

- For the general case, the relation between the DTFTs of the output and the input of a factor-of-*M* down-sampler is given by

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

- $Y(e^{j\omega})$ is a sum of *M* uniformly shifted and stretched versions of $X(e^{j\omega})$ and scaled by a factor of 1/*M*

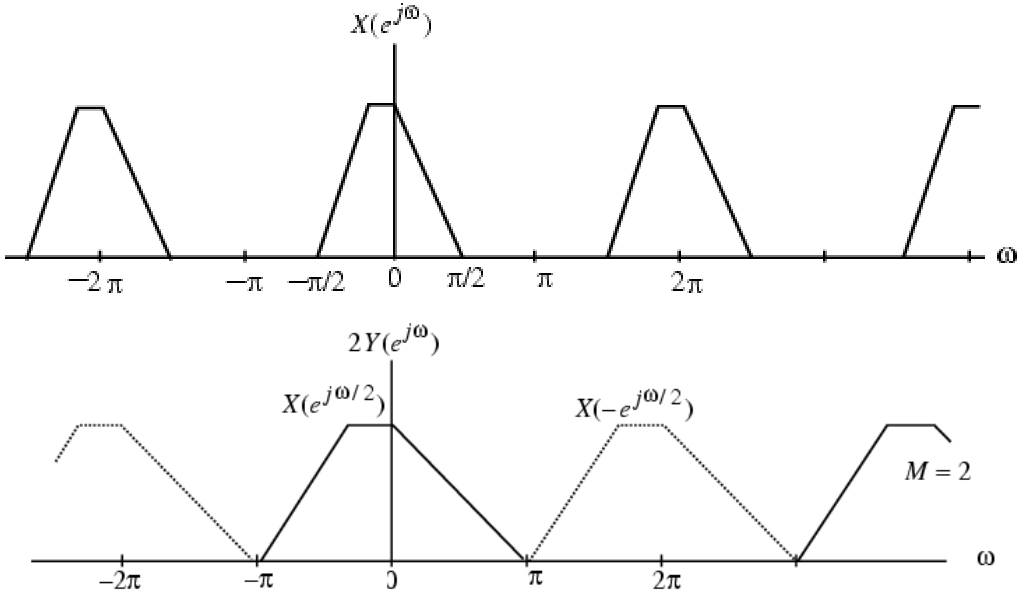
Down-Sampler

- Aliasing is absent if and only if

$$X(e^{j\omega}) = 0 \quad for \; |\omega| \geq \pi / M$$

as shown below for $M = 2$

$$X(e^{j\omega}) = 0 \;\; \text{for} \; |\omega| \geq \pi / 2$$

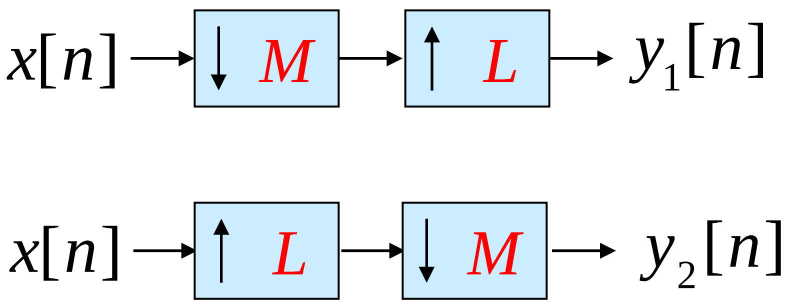


Cascade Equivalences

- A complex ***multirate system*** is formed by an interconnection of the up-sampler, the down-sampler, and the components of an LTI digital filter
- In many applications these devices appear in a cascade form
- An interchange of the positions of the branches in a cascade often can lead to a computationally efficient realization

Cascade Equivalences

- To implement a ***fractional change*** in the ***sampling rate*** we need to employ a cascade of an up-sampler and a down-sampler
- Consider the two cascade connections shown below



Cascade Equivalences

- A cascade of a factor-of-M down-sampler and a factor-of-L up-sampler is interchangeable with no change in the input-output relation:

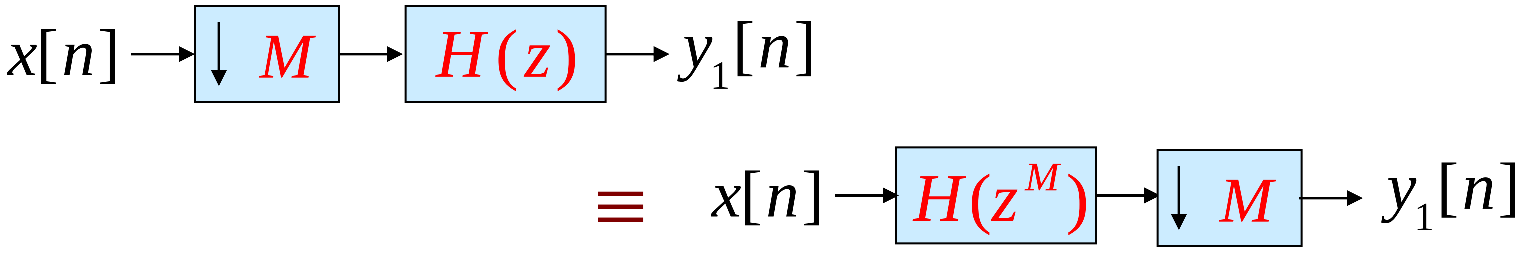
  if and only if M ***and*** L ***are relatively prime***, i.e., M and L do not have any common factor that is an integer $k > 1$

$$y_1[n] = y_2[n]$$

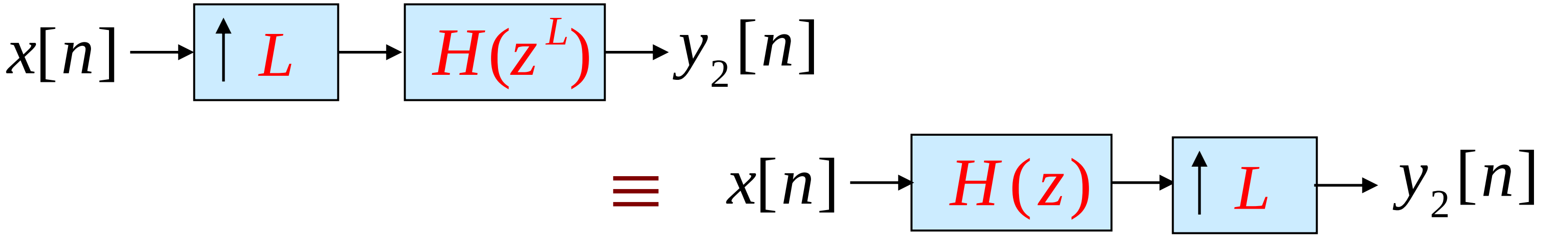
Cascade Equivalences

- Two other cascade equivalences are shown below

Cascade equivalence #1



Cascade equivalence #2



Filters in Sampling Rate Alteration Systems

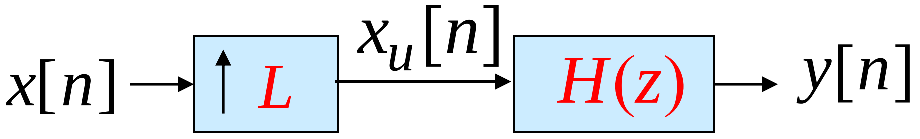
- From the ***sampling theorem*** it is known that a the sampling rate of a critically sampled discrete-time signal with a spectrum occupying the full Nyquist range cannot be reduced any further since such a reduction will introduce aliasing
- Hence, the bandwidth of a critically sampled signal must be reduced by ***lowpass filtering*** before its sampling rate is reduced by a down-sampler

Filters in Sampling Rate Alteration Systems

- Likewise, the zero-valued samples introduced by an up-sampler must be interpolated to more appropriate values for an effective sampling rate increase
- We shall show next that this interpolation can be achieved simply by digital lowpass filtering
- We now develop the frequency response specifications of these lowpass filters

Filter Specifications

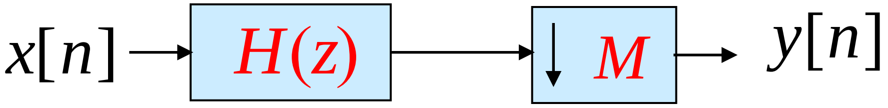
- Since up-sampling causes periodic repetition of the basic spectrum, the unwanted images in the spectra of the up-sampled signal $x_u[n]$ must be removed by using a lowpass filter $H(z)$, called the ***interpolation filter***, as indicated below



- The above system is called an *interpolator*

Filter Specifications

- On the other hand, prior to down-sampling, the signal $v[n]$ should be bandlimited to $|\omega| < \pi / M$ by means of a lowpass filter, called the ***decimation filter***, as indicated below to avoid aliasing caused by down-sampling



- The above system is called a ***decimator***

Interpolation Filter Specifications

- Assume *x*[*n*] has been obtained by sampling a continuous-time signal $x_a(t)$ at the Nyquist rate $X_a(j\Omega)$
- If $X(e^{j\omega})$ and $x_a(t)$ denote the Fourier transforms of and *x*[*n*], respectively, then it can be shown

$$X(e^{j\omega}) = \frac{1}{T_o} \sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi k}{T_o}\right)$$

- where T_o is the sampling period

Interpolation Filter Specifications

- In practice, a transition band is provided to ensure the realizability and stability of the lowpass interpolation filter $H(z)$
- Hence, the desired lowpass filter should have a stopband edge at $\omega_s = \pi / L$ and a passband edge ω_p close to ω_s to reduce the distortion of the spectrum of $x[n]$

Interpolation Filter Specifications

- If ω_c is the highest frequency that needs to be preserved in $x[n]$, then $\omega_p = \omega_c / L$

- Summarizing the specifications of the lowpass interpolation filter are thus given by

$$\left|H(e^{j\omega})\right| = \begin{cases} L, & |\omega| \le \omega_c / L \\ 0, & \pi / L \le |\omega| \le \pi \end{cases}$$

Decimation Filter Specifications

- In a similar manner, we can develop the specifications for the lowpass decimation filter that are given by

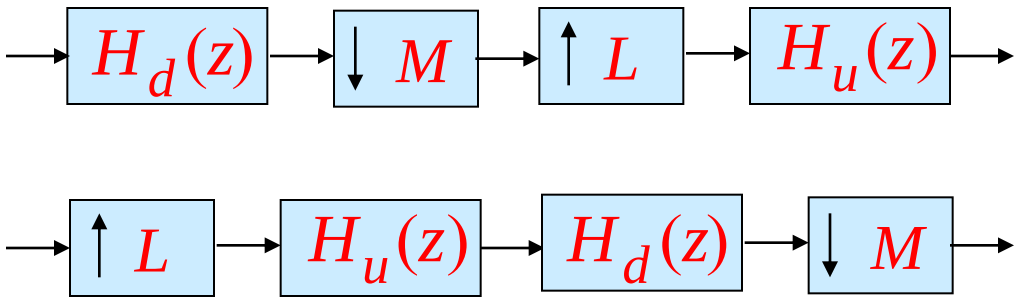
$$\left|H(e^{j\omega})\right| = \begin{cases} 1, & |\omega| \le \omega_c / M \\ 0, & \pi / M \le |\omega| \le \pi \end{cases}$$

Filters for Fractional Sampling Rate Alteration

- A fractional change in the sampling rate can be achieved by cascading a factor-of-M decimator with a factor-of-L interpolator, where M and L are positive integers
- Such a cascade is equivalent to a decimator with a decimation factor of M/L or an interpolator with an interpolation factor of L/M

Filters for Fractional Sampling Rate Alteration

- There are two possible such cascade connections as indicated below



- The second scheme is more computationally efficient since only one of the filters, $H_u(z)$ or $H_d(z)$, is adequate to serve as both the interpolation and the decimation filter

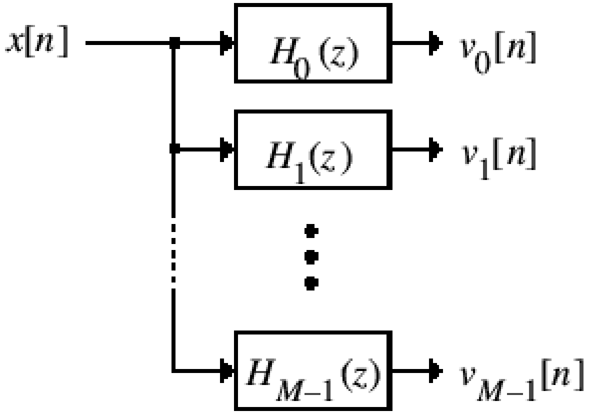
Filters for Fractional Sampling Rate Alteration

- Hence, the desired configuration for the fractional sampling rate alteration is as indicated below where the lowpass filter $H(z)$ has a stopband edge frequency given by

$$\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$

Digital Filter Banks

- The digital filter bank is set of bandpass filters with either a common input or a summed output
- An *M*-band analysis filter bank is shown below

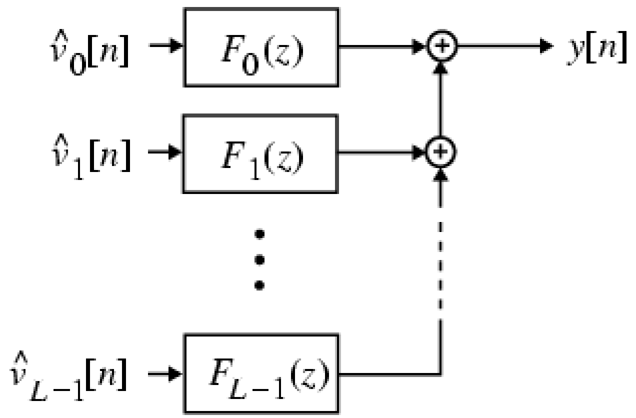


Digital Filter Banks

- The subfilters $H_k(z)$ in the analysis filter bank are known as analysis filters
- The analysis filter bank is used to decompose the input signal $x[n]$ into a set of subband signals $v_k[n]$ with each subband signal occupying a portion of the original frequency band

Digital Filter Banks

- An L-band synthesis filter bank is shown below



- It performs the dual operation to that of the analysis filter bank

Digital Filter Banks

- The subfilters $F_k(z)$ in the synthesis filter bank are known as synthesis filters
- The synthesis filter bank is used to combine a set of subband signals $\hat{v}_k[n]$ (typically belonging to contiguous frequency bands) into one signal $y[n]$ at its output

Uniform Digital Filter Banks

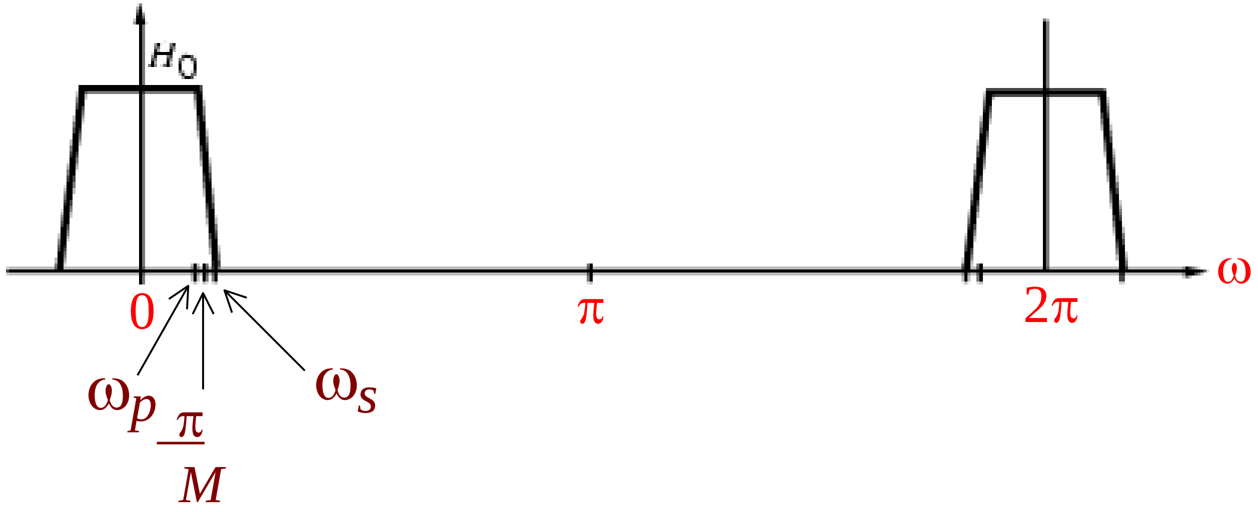
- A simple technique to design a class of filter banks with equal passband widths is outlined next
- Let $H_0(z)$ represent a causal lowpass digital filter with a real impulse response $h_0[n]$:

$$H_0(z) = \sum_{n=-\infty}^{\infty} h_0[n] z^{-n}$$

- The filter $H_0(z)$ is assumed to be an IIR filter without any loss of generality

Uniform Digital Filter Banks

- Assume that $H_0(z)$ has its passband edge ω_p and stopband edge ω_s around π/M, where M is some arbitrary integer, as indicated below



Uniform Digital Filter Banks

- Now, consider the transfer function $H_k(z)$ whose impulse response $h_k[n]$ is given by

$$h_k[n] = h_0[n]e^{j2\pi kn/M} = h_0[n]W_M^{-kn},$$

$$0 \leq k \leq M-1$$

where we have used the notation $W_M = e^{-j2\pi/M}$

- Thus,

$$H_k(z) = \sum_{n=-\infty}^{\infty} h_k[n]z^{-n} = \sum_{n=-\infty}^{\infty} h_0[n]\left(zW_M^k\right)^{-n},$$

$$0 \leq k \leq M-1$$

Uniform Digital Filter Banks

- i.e.,

$$H_k(z) = H_0(zW_M^k), \quad 0 \le k \le M-1$$

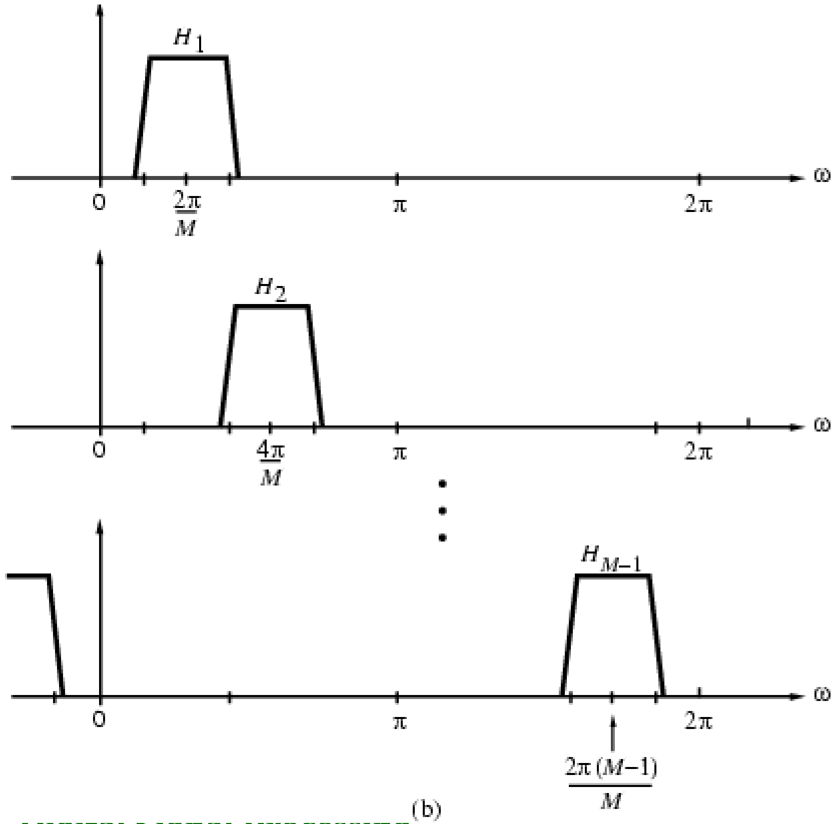
- The corresponding frequency response is given by

$$H_k(e^{j\omega}) = H_0(e^{j(\omega - 2\pi k/M)}), \quad 0 \le k \le M-1$$

- Thus, the frequency response of $H_k(z)$ is obtained by shifting the response of $H_0(z)$ to the right by an amount $2\pi k/M$

Uniform Digital Filter Banks

- The responses of $H_k(z)$, $H_k(z)$, . . . , $H_k(z)$ are shown below



Uniform Digital Filter Banks

- Note: The impulse responses $h_k[n]$ are, in general complex, and hence $|H_k(e^{j\omega})|$ does not necessarily exhibit symmetry with respect to $\omega = 0$
- The responses shown in the figure of the previous slide can be seen to be uniformly shifted version of the response of the basic prototype filter $H_0(z)$

Uniform Digital Filter Banks

- The M filters defined by

$$H_k(z) = H_0(zW_M^k), \quad 0 \le k \le M-1$$

could be used as the analysis filters in the analysis filter bank or as the synthesis filters in the synthesis filter bank

- Since the magnitude responses of all M filters are uniformly shifted version of that of the prototype filter, the filter bank obtained is called a uniform filter bank

Uniform DFT Filter Banks

Polyphase Implementation

- Let the prototype lowpass transfer function be represented in its M-band polyphase form:

$$H_0(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$$

where $E_\ell(z)$ is the ℓ-th polyphase component of $H_0(z)$:

$$E_\ell(z) = \sum_{n=0}^{\infty} e_\ell[n] z^{-n} = \sum_{n=0}^{\infty} h_0[\ell + nM] z^{-n},$$

$$0 \le \ell \le M-1$$

Uniform DFT Filter Banks

- Substituting z with zW_M^k in the expression for $H_0(z)$ we arrive at the M-band polyphase decomposition of $H_k(z)$:

$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} W_M^{-k\ell} E_\ell(z^M W_M^{kM})$$

$$= \sum_{\ell=0}^{M-1} z^{-\ell} W_M^{-k\ell} E_\ell(z^M), \quad 0 \le k \le M-1$$

- In deriving the last expression we have used the identity $W_M^{kM} = 1$

Uniform DFT Filter Banks

- The equation on the previous slide can be written in matrix form as

$$H_k(z) = [1 \quad W_M^{-k} \quad W_M^{-2k} \quad \cdots \quad W_M^{-(M-1)k}] \begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ z^{-2}E_2(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix}$$

$$0 \le k \le M-1$$

Uniform DFT Filter Banks

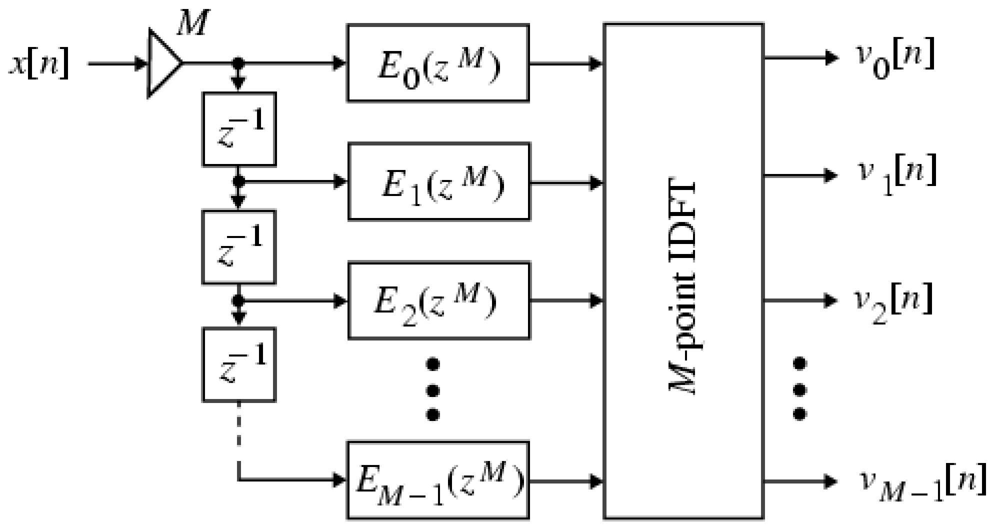
- All M equations on the previous slide can be combined into one matrix equation as

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)} \\ 1 & W_M^{-2} & W_M^{-4} & \cdots & W_M^{-2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)^2} \end{bmatrix}}_{M\mathbf{D}^{-1}} \begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ z^{-2}E_2(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix}$$

- In the above $\mathbf{D}$ is the $M \times M$ DFT matrix

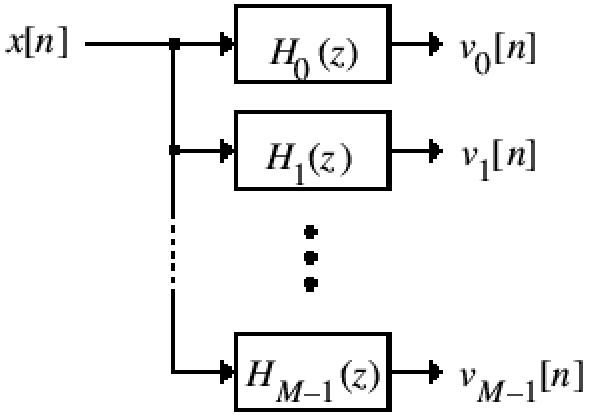
Uniform DFT Filter Banks

- An efficient implementation of the M-band uniform analysis filter bank, more commonly known as the uniform DFT analysis filter bank, is then as shown below



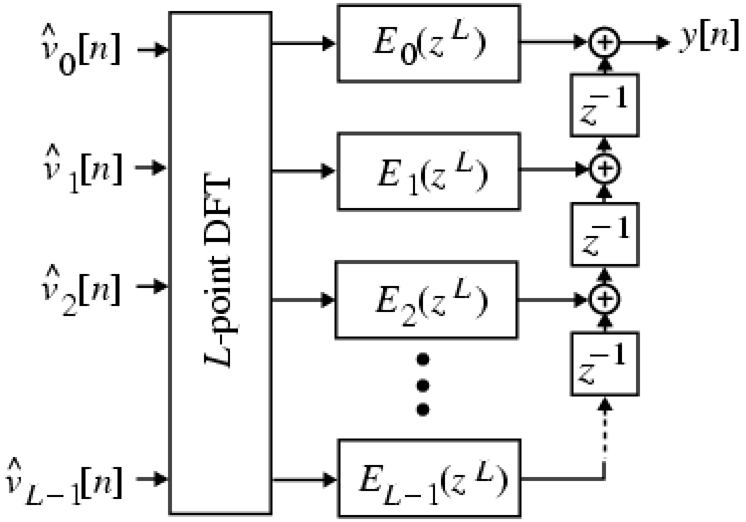
Uniform DFT Filter Banks

- The computational complexity of an M-band uniform DFT filter bank is much smaller than that of a direct implementation as shown below

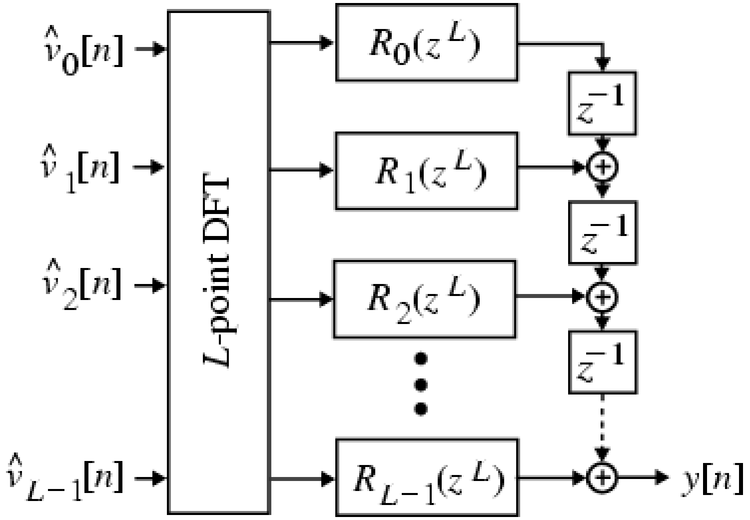


Uniform DFT Filter Banks

- Following a similar development, we can derive the structure for a uniform DFT synthesis filter bank as shown below



Type I uniform DFT synthesis filter bank



Type II uniform DFT

Uniform DFT Filter Banks

- Now $E_i(z^M)$ can be expressed in terms of

$$\begin{bmatrix} E_0(z^M) \\ z^{-1}E_1(z^M) \\ z^{-2}E_2(z^M) \\ \vdots \\ z^{-(M-1)}E_{M-1}(z^M) \end{bmatrix} = \frac{1}{M}\,\mathbf{D}\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}$$

- The above equation can be used to determine the polyphase components of an IIR transfer function $H_0(z)$