ECE 301: Signals and Systems

Course Notes

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Acknowledgments

These notes very closely follow the book: *Signals and Systems*, 2nd edition, by Alan V. Oppenheim, Alan S. Willsky with S. Hamid Nawab. Parts of the notes are also drawn from

- *Linear Systems and Signals* by B. P. Lathi
- *A Course in Digital Signal Processing* by Boaz Porat
- *Calculus for Engineers* by Donald Trim

I claim credit for all typos and mistakes in the notes.

The LaTeX template for *The Not So Short Introduction to LaTeX 2_ε* by T. Oetiker et al. was used to typeset portions of these notes.

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Chapter 1

Introduction

1.1 Signals and Systems

Loosely speaking, *signals* represent information or data about some phenomenon of interest. This is a very broad definition, and accordingly, signals can be found in every aspect of the world around us.

For the purposes of this course, a *system* is an abstract object that accepts *input signals* and produces *output signals* in response.

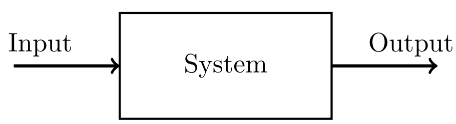

Figure 1.1: An abstract representation of a system.

Examples of systems and associated signals:

- Electrical circuits: voltages, currents, temperature,...
- Mechanical systems: speeds, displacement, pressure, temperature, volume, ...
- Chemical and biological systems: concentrations of cells and reactants, neuronal activity, cardiac signals, ...
- Environmental systems: chemical composition of atmosphere, wind patterns, surface and atmospheric temperatures, pollution levels, ...
- Economic systems: stock prices, unemployment rate, tax rate, interest rate, GDP, ...
- Social systems: opinions, gossip, online sentiment, political polls,...
- Audio/visual systems: music, speech recordings, images, video, ...

- Computer systems: Internet traffic, user input, ...

From a mathematical perspective, signals can be regarded as functions of one or more independent variables. For example, the voltage across a capacitor in an electrical circuit is a function of time. A static monochromatic image can be viewed as a function of two variables: an x-coordinate and a y-coordinate, where the value of the function indicates the brightness of the pixel at that (x, y) coordinate. A video is a sequence of images, and thus can be viewed as a function of three variables: an x-coordinate, a y-coordinate and a time-instant. Chemical concentrations in the earth's atmosphere can also be viewed as functions of space and time.

In this course, we will primarily be focusing on signals that are functions of a single independent variable (typically taken to be time). Based on the examples above, we see that this class of signals can be further decomposed into two subclasses:

- A *continuous-time signal* is a function of the form $f(t)$, where t ranges over all real numbers (i.e., $t \in \mathbb{R}$).
- A *discrete-time signal* is a function of the form $f[n]$, where n takes on only a discrete set of values (e.g., $n \in \mathbb{Z}$).

Note that we use square brackets to denote discrete-time signals, and round brackets to denote continuous-time signals. Examples of continuous-time signals often include physical quantities, such as electrical currents, atmospheric concentrations and phenomena, vehicle movements, etc. Examples of discrete-time signals include the closing prices of stocks at the end of each day, population demographics as measured by census studies, and the sequence of frames in a digital video. One can obtain discrete-time signals by *sampling* continuous-time signals (i.e., by selecting only the values of the continuous-time signal at certain intervals).

Just as with signals, we can consider continuous-time systems and discrete-time systems. Examples of the former include atmospheric, physical, electrical and biological systems, where the quantities of interest change continuously over time. Examples of discrete-time systems include communication and computing systems, where transmissions or operations are performed in scheduled time-slots. With the advent of ubiquitous sensors and computing technology, the last few decades have seen a move towards *hybrid* systems consisting of both continuous-time and discrete-time subsystems – for example, digital controllers and actuators interacting with physical processes and infrastructure. We will not delve into such hybrid systems in this course, but will instead focus on systems that are entirely either in the continuous-time or discrete-time domain.

The term *dynamical system* loosely refers to any system that has an internal state and some dynamics (i.e., a rule specifying how the state evolves in time).

This description applies to a very large class of systems, including individual vehicles, biological, economic and social systems, industrial manufacturing plants, electrical power grid, the state of a computer system, etc. The presence of dynamics implies that the behavior of the system cannot be entirely arbitrary; the temporal behavior of the system's state and outputs can be predicted to some extent by an appropriate *model* of the system.

Example 1.1. Consider a simple model of a car in motion. Let the speed of the car at any time t be given by $v(t)$. One of the inputs to the system is the acceleration $a(t)$, applied by the throttle. From basic physics, the evolution of the speed is given by

$$\frac{dv}{dt} = a(t). \tag{1.1}$$

The quantity $v(t)$ is the state of the system, and equation (1.1) specifies the dynamics. There is a speedometer on the car, which is a sensor that measures the speed. The value provided by the sensor is denoted by $s(t) = v(t)$, and this is taken to be the output of the system. □

Much of scientific and engineering endeavor relies on gathering, manipulating and understanding signals and systems across various domains. For example, in communication systems, the signal represents voice or data that must be transmitted from one location to another. These information signals are often corrupted en route by other noise signals, and thus the received signal must be processed in order to recover the original transmission. Similarly, social, physical and economic signals are of great value in trying to predict the current and future state of the underlying systems. The field of signal processing studies how to take given signals and extract desirable features from them, often via the design of systems known as *filters*. The field of control systems focuses on designing certain systems (known as controllers) that measure the signals coming from a given system and apply other input signals in order to make the given system behave in an desirable manner. Typically, this is done via a *feedback loop* of the form

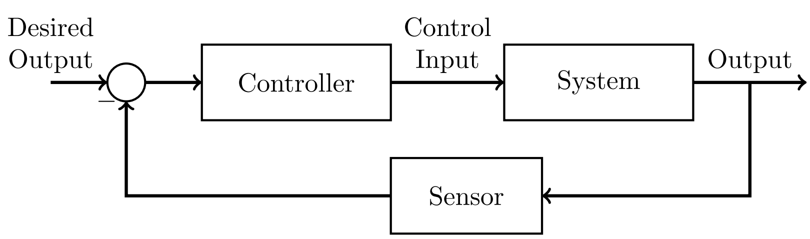

Figure 1.2: Block Diagram of a feedback control system.

Example 1.2 (Inverted Pendulum)**.** Suppose we try to balance a stick vertically in the palm of our hand. The sensor, controller and actuator in this example are our eyes, our brain, and our hand, respectively, which communicate using signals of various forms. This is an example of a feedback control system. □

1.2 Outline of This Course

Since the concepts of signals and systems are prevalent across a wide variety of domains, we will not attempt to discuss each specific application in this course. Instead, we will deal with the underlying mathematical theory, analysis, and design of signals and systems. In this sense, it will be more mathematical than other engineering courses, but will be different from other math courses in that it will pull together various branches of mathematics for a particular purpose (i.e., to understand the nature of signals and systems).

The main components of this course will be as follows.

- Signal and systems classifications: develop terminology and identify useful properties of signals and systems
- Time domain analysis of LTI systems: understand how the output of linear time-invariant systems is related to the input
- Frequency domain analysis techniques and signal transformations (Fourier, Laplace, z-transforms): study methods to study signals and systems from a frequency domain perspective, gaining new ways to understand their behavior
- Sampling and Quantization: study ways to convert continuous-time signals into discrete-time signals, along with associated challenges

The material in this course will lay the foundations for future courses in control theory (ECE 382, ECE 483), communication systems (ECE 440) and signal processing (ECE438, 445).

Chapter 2

Properties of Signals and Systems

We will now identify certain useful properties and classes of signals and systems. Recall that a continuous-time signal is denoted by $f(t)$ (i.e., a function of the real-valued variable t) and a discrete-time signal is denoted by $f[n]$ (i.e., a function of the integer-valued variable n). When drawing discrete-time signals, we will use a sequence of dots to indicate the discrete nature of the time variable.

2.1 Signal Energy and Power

Suppose that we consider a resistor in an electrical circuit, and let $v(t)$ denote the voltage signal across the resistor $i(t)$ denote the current. From Ohm's law, we know that $v(t) = i(t)R$, where R is the resistance. The *power* dissipated by the resistor is then

$$p(t) = v(t)i(t) = i^2(t)R = \frac{v^2(t)}{R}.$$

Thus the power is a scaled multiple of the *square* of the voltage and current signals.

Since the *energy* expended over a time-interval $[t_1, t_2]$ is given by the integral of the power dissipated per-unit-time over that interval, we have

$$E = \int_{t_1}^{t_2} p(t)dt = R\int_{t_1}^{t_2} i^2(t)dt = \frac{1}{R}\int_{t_1}^{t_2} v^2(t)dt.$$

The *average power* dissipated over the time-interval $[t_1, t_2]$ is then

$$\frac{1}{t_2 - t_1}E = \frac{1}{t_2 - t_1}\frac{1}{R}\int_{t_1}^{t_2} v^2(t)dt = \frac{R}{t_2 - t_1}\int_{t_1}^{t_2} i^2(t)dt.$$

We will find it useful to discuss the energy and average power of any continuous-time or discrete-time signal. In particular, the *energy* of a general (potentially complex-valued) continuous-time signal $f(t)$ over a time-interval $[t_1, t_2]$ is defined as

$$E_{[t_1,t_2]} \triangleq \int_{t_1}^{t_2} |f(t)|^2 dt,$$

where $|f(t)|$ denotes the magnitude of the signal at time t.

Similarly, the energy of a general (potentially complex-valued) discrete-time signal $f[n]$ over a time-interval $[n_1, n_2]$ is defined as

$$E_{[n_1,n_2]} \triangleq \sum_{n=n_1}^{n_2} |f[n]|^2.$$

Note that we are *defining* the energy of an arbitrary signal in the above way; this will end up being a convenient way to measure the "size" of a signal, and may not actually correspond to any physical notion of energy.

We will also often be interested in measuring the energy of a given signal over all time. In this case, we define

$$E_\infty \triangleq \int_{-\infty}^{\infty} |f(t)|^2 dt$$

for continuous-time signals, and

$$E_\infty \triangleq \sum_{n=\infty}^{\infty} |f[n]|^2$$

for discrete-time signals. Note that the quantity E_∞ may not be finite.

Similarly, we define the *average power* of a continuous-time signal as

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt,$$

and for a discrete-time signal as

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |f[n]|^2.$$

Based on the above definitions, we have three classes of signals: finite energy ($E_\infty < \infty$), finite average power ($P_\infty < \infty$), and those that have neither finite energy nor average power. An example of the first class is the signal $f(t) = e^{-t}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. An example of the second class is $f(t) = 1$ for all $t \in \mathbb{R}$. An example of the third class is $f(t) = t$ for $t \geq 0$. Note that any signal that has finite energy will also have finite average power, since

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

for continuous-time signals with finite energy, with an analogous characterization for discrete-time signals.

2.2 Transformations of Signals

Throughout the course, we will be interested in manipulating and transforming signals into other forms. Here, we start by considering some very simple transformations involving the time variable. For the purposes of introducing these transformations, we will consider a continuous-time signal $f(t)$ and a discrete-time signal $f[n]$.

Time-shifting: Suppose we define another signal $g(t) = f(t-t_0)$, where $t_0 \in \mathbb{R}$. In other words, for every $t \in \mathbb{R}$, the value of the signal $g(t)$ at time t is the value of the signal $f(t)$ at time $t - t_0$. If $t_0 > 0$, then $g(t)$ is a "forward-shifted" (or time-delayed) version of $f(t)$, and if $t_0 < 0$, then $g(t)$ is a time-advanced version of $f(t)$ (i.e., the features in $f(t)$ appear earlier in time in $g(t)$). Similarly, for a discrete-time signal $f[n]$, one can define the time-shifted signal $f[n-n_0]$, where n_0 is some integer.

Time-reversal: Consider the signal $g(t) = f(-t)$. This represents a reversal of the function $f(t)$ in time. Similarly, $f[-n]$ represents a time-reversed version of the signal $f[n]$.

Time-scaling: Define the signal $g(t) = f(\alpha t)$, where α is some real number. When $0 < \alpha < 1$, this represents a *stretching* of $f(t)$, and when $\alpha > 1$, this represents a *compression* of $f(t)$. If $\alpha < 0$, we get a time-reversed and stretched (or compressed) version of $f(t)$. Analogous definitions hold for the discrete-time signal $f[n]$.

The operations above can be combined to define signals of the form $g(t) = f(\alpha t + \beta)$, where α and β are real numbers. To draw the signal $g(t)$, we should first apply the time-shift by β to $f(t)$ and *then* apply the scaling α. To see why, define $h(t) = f(t + \beta)$, and $g(t) = h(\alpha t)$. Thus, we have $g(t) = f(\alpha t + \beta)$ as required. If we applied the operations in the other order, we would first get the signal $h(t) = f(\alpha t)$, and then $g(t) = h(t + \beta) = f(\alpha(t + \beta)) = f(\alpha t + \alpha\beta)$. In other words, the shift would be by $\alpha\beta$ rather than β.

Examples of these operations can be found in the textbook (OW), such as example 1.1.

2.3 Periodic, Even and Odd Signals

A continuous-time signal $f(t)$ is said to be *periodic* with period T if $f(t) = f(t + T)$ for all $t \in \mathbb{R}$. Similarly, a discrete-time signal $f[n]$ is periodic with period N if $f[n] = f[n + N]$ for all $n \in \mathbb{Z}$. The *fundamental period* of a signal is the smallest period for which the signal is periodic.

A signal is *even* if $f(t) = f(-t)$ for all $t \in \mathbb{R}$ (in continuous-time), or $f[n] = f[-n]$ for all $n \in \mathbb{Z}$ (in discrete-time). A signal is *odd* if $f(t) = -f(-t)$ for all $t \in \mathbb{R}$, or $f[n] = -f[-n]$ for all $n \in \mathbb{Z}$. Note that if a signal is odd, it must necessarily be zero at time 0 (since $f(0) = -f(0)$).

Given a signal $f(t)$, define the signals

$$e(t) = \frac{1}{2}\left(f(t) + f(-t)\right), \quad o(t) = \frac{1}{2}\left(f(t) - f(-t)\right).$$

It is easy to verify that $o(t)$ is an odd signal and $e(t)$ is an even signal. Furthermore, $x(t) = e(t) + o(t)$. Thus, any signal can be decomposed as a sum of an even signal and an odd signal.

2.4 Exponential and Sinusoidal Signals

2.4.1 Continuous-Time Complex Exponential Signals

Consider a signal of the form

$$f(t) = Ce^{at}$$

where C and a are complex numbers. If both C and a are real, there are three possible behaviors for this signal. If $a < 0$, then the signal goes to zero as $t \to \infty$, and if $a > 0$, the signal goes to ∞ as $t \to \infty$. For $a = 0$, the signal is constant.

Now suppose $f(t) = e^{j(\omega_0 t + \phi)}$ for some positive real number ω_0 and real number ϕ (this corresponds to $C = e^{j\phi}$ and $a = j\omega_0$ in the signal given above). We first note that

$$f(t+T) = e^{j(\omega_0(t+T)+\phi)} = e^{j(\omega_0 t+\phi)}e^{j\omega_0 T}.$$

If T is such that $\omega_0 T$ is an integer multiple of 2π, we have $e^{j\omega_0 T} = 1$ and the signal is periodic with period T. Thus, the fundamental period of this signal is

$$T_0 = \frac{2\pi}{\omega_0}.$$

Note that if $\omega_0 = 0$, then $f(t) = 1$ and is thus periodic with any period. The fundamental period is undefined in this case. Also note that $f(t) = e^{-j\omega_0 T}$ is also periodic with period T_0. The quantity ω_0 is called the *fundamental frequency* of the signal.

Note that periodic signals (other than the one that is zero for all time) have infinite energy, but finite average power. Specifically, let

$$E_p = \int_0^{T_0} |f(t)|^2 dt$$

be the energy of the signal over one period. The average power over that period is then $P_p = \frac{E_p}{T_0}$, and since this extends over all time, this ends up being the average power of the signal as well. For example, for the signal $f(t) = e^{j(\omega_0 t+\phi)}$, we have

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} 1dt = 1.$$

Given a complex exponential with fundamental frequency ω_0, a *harmonically related set* of complex exponentials is a set of periodic exponentials of the form

$$\phi_k(t) = e^{jk\omega_0 t}, k \in \mathbb{Z}.$$

In other words, it is the set of complex exponentials whose frequencies are multiples of the fundamental frequency ω_0. Note that if $e^{j\omega_0 t}$ is periodic with period T_0 (i.e, $\omega_0 T_0 = 2\pi m$ for some integer m), then $\phi_k(t)$ is also periodic with period T_0 for any $k \in \mathbb{Z}$, since

$$\phi_k(t+T_0) = e^{jk\omega_0(t+T_0)} = e^{jk\omega_0 T_0}e^{jk\omega_0 t} = e^{jkm2\pi}\phi_k(t) = \phi_k(t).$$

Although the signal $f(t)$ given above is complex-valued in general, its real part and imaginary part are sinusoidal. To see this, use Euler's formula to obtain

$$Ae^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t+\phi) + jA\sin(\omega_0 t+\phi).$$

Similarly, we can write

$$A\cos(\omega_0 t+\phi) = \frac{A}{2}e^{j(\omega_0 t+\phi)} + \frac{A}{2}e^{-j(\omega_0 t+\phi)},$$

i.e., a sinusoid can be written as a sum of two complex exponential signals.

Using the above, there are two main observations. First, continuous-time complex exponential signals are periodic for any $\omega_0 \in \mathbb{R}$ (the fundamental period is $\frac{2\pi}{\omega_0}$ for $\omega_0 \neq 0$ and undefined otherwise). Second, the larger ω_0 gets, the smaller the period gets.

We will now look at discrete-time complex exponential signals and see that the above two observations do not necessarily hold for such signals.

2.4.2 Discrete-Time Complex Exponential Signals

As in the continuous-time case, a discrete-time complex exponential signal is of the form

$$f[n] = Ce^{an}$$

where C and a are general complex numbers. As before, let us focus on the case where $C = 1$ and $a = j\omega_0$ for some $\omega_0 \in \mathbb{R}$ in order to gain some intuition, i.e., $f[n] = e^{j\omega_0 n}$.

To see the differences in discrete-time signals from continuous-time signals, recall that a continuous-time complex exponential is always periodic for any ω_0. The first difference between discrete-time complex exponentials and continuous-time complex exponentials is that discrete-time complex exponentials are not necessarily periodic. Specifically, consider the signal $f[n] = e^{j\omega_0 n}$, and suppose it is periodic with some period N_0. Then by definition, it must be the case that

$$f[n+N_0] = e^{j\omega_0(n+N_0)} = e^{j\omega_0 n}e^{j\omega_0 N_0} = f[n]e^{j\omega_0 N_0}.$$

Due to periodicity, we must have $e^{j\omega_0 N_0} = 1$, or equivalently, $\omega_0 N_0 = 2\pi k$ for some integer k. However, N_0 must be an integer, and thus we see that this can be satisfied if and only of ω_0 is a *rational multiple of* 2π. In other words, only discrete-time complex exponentials whose frequencies are of the form

$$\omega_0 = 2\pi \frac{k}{N}$$

for some integers k and N are periodic. The *fundamental period* N_0 of a signal is the smallest nonnegative integer for which the signal is periodic. Thus, for discrete-time complex exponentials, we find the fundamental period by first writing

$$\frac{\omega_0}{2\pi} = \frac{k}{N}$$

where k and N have no factors in common. The value of N in this representation is then the fundamental period.

Example 2.1. Consider the signal $f[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$. Since both of the exponentials have frequencies that are rational multiples of 2π, they are both periodic. For the first exponential, we have

$$\frac{\frac{2\pi}{3}}{2\pi} = \frac{1}{3},$$

which cannot be reduced any further. Thus the fundamental period of the first exponential is 3. Similarly, for the second exponential, we have

$$\frac{\frac{3\pi}{4}}{2\pi} = \frac{3}{8}.$$

Thus the fundamental period of the second exponential is 8. Thus $f[n]$ is periodic with period 24 (the least common multiple of the periods of the two signals). □

The same reasoning applies to sinusoids of the form $f[n] = \cos(\omega_0 n)$. A necessary condition for this function to be periodic is that there are two positive integers n_1, n_2 with $n_2 > n_1$ such that $f[n_1] = f[n_2]$. This is equivalent to $\cos(\omega_0 n_1) = \cos(\omega_0 n_2)$. Thus, we must either have

$$\omega_0 n_2 = \omega_0 n_1 + 2\pi k$$

or

$$\omega_0 n_2 = -\omega_0 n_1 + 2\pi k$$

for some positive integer k. In either case, we see that ω_0 has to be a rational multiple of 2π. In fact, when ω_0 is not a rational multiple of 2π, the function $\cos(\omega_0 n)$ *never* takes the same value twice for positive values of n.

The second difference from continuous-time complex exponentials pertains to the period of oscillation. Specifically, even for periodic discrete-time complex exponentials, increasing the frequency does not necessarily make the period smaller. Consider the signal $g[n] = e^{j(\omega_0+2\pi)n}$, i.e., a complex exponential with frequency $\omega_0 + 2\pi$. We have

$$g[n] = e^{j\omega_0 n}e^{j2\pi n} = e^{j\omega_0 n} = f[n],$$

i.e., the discrete-time complex exponential with frequency $\omega_0 + 2\pi$ is the same as the discrete-time complex exponential with frequency ω_0, and thus they have the same fundamental period. More generally, any two complex exponential signals whose frequencies differ by an integer multiple of 2π are, in fact, the same signal.

This shows that all of the unique complex exponential signals of the form $e^{j\omega_0 n}$ have frequencies that are confined to a region of length 2π. Typically, we will consider this region to be $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$. Suppose we consider the interval $0 \le \omega_0 < 2\pi$. Note that

$$e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n).$$

As ω_0 increases from 0 to π, the frequencies of both the sinusoids increase.[1] For those ω_0 between 0 and π that are also rational multiples of π, the sampled signals will be periodic, and their period will decrease as ω_0 increases.

Now suppose $\pi \le \omega_0 < 2\pi$. Consider the frequency $2\pi - \omega_0$, which falls between 0 and π. We have

$$e^{j(2\pi-\omega_0)n} = e^{-j\omega_0 n} = \cos(\omega_0 n) - j\sin(\omega_0 n).$$

Since $e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$, the frequency of oscillation of the discrete-time complex exponential with frequency ω_0 is the same as the frequency of oscillation of the discrete-time complex exponential with frequency $2\pi - \omega_0$. Thus, as ω_0 crosses π and moves towards 2π, the frequency of oscillation starts to decrease.

To illustrate this, it is again instructive to consider the sinusoidal signals $f[n] = \cos(\omega_0 n)$ for $\omega_0 \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. When $\omega_0 = 0$, the function is simply constant at 1 (and thus its period is undefined). We see that the functions with $\omega_0 = \frac{\pi}{2}$ and $\omega_0 = \frac{3\pi}{2}$ have the same period (in fact, they are exactly the same function).[2]

The following table shows the differences between continuous-time and discrete-time signals.

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[1] As we will see later in the course, the signals $\cos(\omega_0 n)$ and $\sin(\omega_0 n)$ correspond to continuous-time signals of the form $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ that are sampled at 1 Hz. When $0 \le \omega_0 < \pi$, this sampling rate is above the Nyquist frequency $\frac{\omega_0}{\pi}$, and thus the sampled signals will be an accurate representation of the underlying continuous-time signal.

[2] Note that $\cos(\omega_0 n) = \cos((2\pi - \omega_0)n)$ for any $0 \le \omega_0 \le \pi$. The same is not true for $\sin(\omega_0 n)$. In fact, one can show that for any two different frequencies $0 \le \omega_0 < \omega_1 \le 2\pi$, $\sin(\omega_0 n)$ and $\sin(\omega_1 n)$ are different functions.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for different values of ω_0	Identical signals for values of ω_0 separated by 2π
Periodic for any ω_0	Periodic only if $\omega_0 = 2\pi\frac{k}{N}$ for some integers k and $N > 0$.
Fundamental period: undefined for $\omega_0 = 0$ and $\frac{2\pi}{\omega_0}$ otherwise	Fundamental period undefined for $\omega_0 = 0$ and $k\frac{2\pi}{\omega_0}$ otherwise
Fundamental frequency ω_0	Fundamental frequency $\frac{\omega_0}{k}$

As with continuous-time signals, for any period N, we define the harmonic family of discrete-time complex exponentials as

$$\phi_k[n] = e^{jk\frac{2\pi}{N}n}, \quad k \in \mathbb{Z}.$$

This is the set of all discrete-time complex exponentials that have a common period N, and frequencies whose multiples of $\frac{2\pi}{N}$. This family will play a role in our analysis later in the course.

2.5 Impulse and Step Functions

2.5.1 Discrete-Time

The **discrete-time unit impulse** signal (or function) is defined as

$$\delta[n] = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}.$$

The **discrete-time unit step** signal is defined as

$$u[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}.$$

Note that by the time-shifting property, we have

$$\begin{aligned} \delta[n] &= u[n] - u[n-1] \\ u[n] &= \sum_{k=0}^{\infty} \delta[n-k]. \end{aligned}$$

In other words, the unit step function can be viewed as a superposition of shifted impulse functions.

Suppose we are given some arbitrary signal $f[n]$. If we multiply $f[n]$ by the time-shifted impulse function $\delta[n-k]$, we get a signal that is zero everywhere except at $n = k$, where it takes the value $f[k]$. This is known as the *sampling* or *sifting* property of the impulse function:

$$f[n]\delta[n-k] = f[k]\delta[n-k].$$

More generally, for any signal $f[n]$, we have

$$f[n] = \sum_{k=-\infty}^{\infty} f[k]\delta[n-k],$$

i.e., any function $f[n]$ can be written as a sum of scaled and shifted impulse functions.

2.5.2 Continuous-Time

The **continuous-time unit step function** is defined by

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

Note that $u(t)$ is discontinuous at $t = 0$ (we will take it to be continuous from the right).

To define the continuous-time analog of the discrete-time impulse function, we first define the signal

$$\delta_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\epsilon} & \text{if } 0 \leq t \leq \epsilon \\ 0 & \text{if } t > \epsilon \end{cases},$$

where $\epsilon \in \mathbb{R}_{>0}$. Note that for any $\epsilon > 0$, we have $\int_{-\infty}^{\infty} \delta_\epsilon(t)dt = 1$. As ϵ gets smaller, the width of this function gets smaller and the height increases proportionally. The continuous-time impulse function is defined as the limit of the above function as ϵ approaches zero from the right:

$$\delta(t) = \lim_{\epsilon \downarrow 0} \delta_\epsilon(t).$$

This function is drawn with an arrow at the origin (since it has no width and infinite height). We will often be interested in working with scaled and time-shifted versions of the continuous-time impulse function. Just as we did with discrete-time functions, we can take a continuous-time function $f(t)$ and represent it as

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau.$$

In other words, if we take an infinite sequence of shifted impulse functions, scale each of them by the value of the function $f(t)$ at the value of the time-shift, and add them together (represented by the integration), we get the function $f(t)$. For instance, we have

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau.$$

Just as the discrete-time impulse function could be viewed as a difference of the discrete-time unit step and its time-shifted version, the continuous-time impulse function can be viewed as the derivative of the continuous-time unit step function.

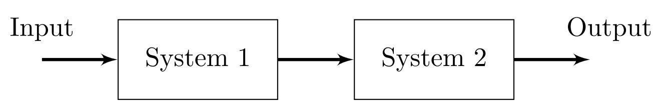
2.6 Properties of Systems

As we discussed during the first lecture, a system can be viewed as an abstract object that takes input signals and produces output signals. A continuous-time system operates on continuous-time signals, and discrete-time systems operate with discrete-time signals. Examples of the former include many physical systems such as electrical circuits, vehicles, etc. Examples of the latter include computer systems, a bank account where the amount of money is incremented with interest, deposits and withdrawals at the end of each day, etc.

2.6.1 Interconnections of Systems

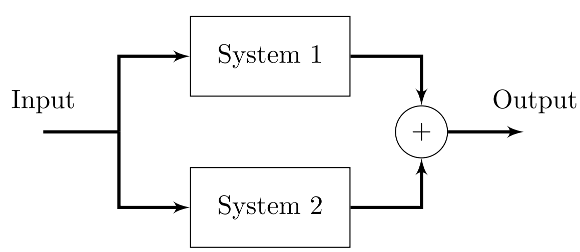
We will often be interested in connecting different systems together in order to achieve a certain objective. There are three basic interconnection patterns that are used to build more complicated interconnections.

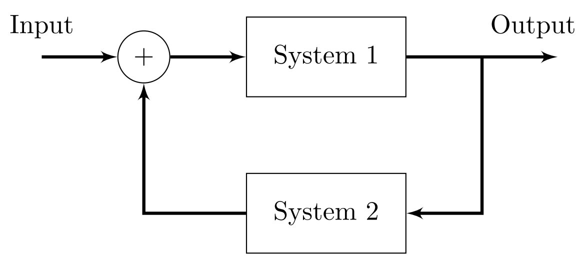
The first is a **serial** connection of systems:



An example of a series interconnection of systems occurs in communication systems; the signal to be transmitted is first passed through an *encoder*, which transforms the signal into a form suitable for transmission. That transformed signal is then sent through the communication channel (the second system in the chain). The output of the communication channel is then passed through a *decoder*, which is the third system in the chain. The output of the decoder is an estimate of the signal that entered the encoder.

The second type of interconnection is a **parallel** interconnection:





The third type of interconnection is a **feedback** interconnection:

Feedback interconnections form the basis of control systems; in this case we are given a specific system (System 1) that we wish to control (or make behave in a certain way). The second system (System 2) is a *controller* that we design in order to achieve the desired behavior. This controller takes the current output of the system and uses that to decide what other inputs to apply to the original system in order to change the output appropriately.

2.6.2 Properties of Systems

Systems With and Without Memory

A system is **memoryless** if the output each time-instant (either in discrete-time or continuous-time) only depends on the input at that time-instant. For example, the system

$$y[n] = \cos(x[n])$$

is memoryless, as the output at each time-step $n \in \mathbb{Z}$ only depends on the input at that time-step.

However, the system whose input and output are related by

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

is not memoryless, as the output depends on all of the input values from the past.

Systems with memory are often represented as having some sort of *state* and *dynamics*, which maintains the necessary information from the past. For example, for the system given above, we can use the fundamental theorem of calculus to obtain

$$\frac{dy(t)}{dt} = x(t)$$

where the state of the system is $y(t)$ (this is also the output), and the dynamics of the state are given by the differential equation above. Similarly for the discrete-time system

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

we have

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n]$$

which is a *difference equation* describing how the state $y[n]$ evolves over time.

Invertibility

A system is said to be **invertible** if distinct inputs lead to distinct outputs. In other words, by looking at the output of a system, one can uniquely identify what the input was.

An example of an invertible system is $y(t) = \alpha x(t)$, where α is any nonzero real number. Given the output $y(t)$, we can uniquely identify the input as $x(t) = \frac{1}{\alpha} y(t)$. However, if $\alpha = 0$, then we have $y(t) = 0$ regardless of the input, and there is no way to recover the input. In that case, the system would not be invertible.

Another example of a noninvertible system is $y(t) = x^2(t)$, as the sign of the input is lost when converting to the output.

The system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is invertible; to see this, we use the equivalent representation $y[n] = y[n-1] + x[n]$ to obtain $x[n] = y[n] - y[n-1]$ for all $n \in \mathbb{Z}$.

Causality

A system is **causal** if the output of the system at any time depends only on the input at that time and from the past. In other words, for all $t \in \mathbb{R}$, $y(t)$ depends only on $x(\tau)$ for $\tau \leq t$. Thus, a causal system does not react to inputs that will happen in the future. For a causal system, if two different inputs have the same values up to a certain time, the output of the system due to those two inputs will agree up to that time as well. All memoryless systems are causal.

There are various instances where we may wish to use noncausal systems. For example, if we have time-series data saved offline, we can use the saved values of the signal for $k > n$ to process the signal at a given time-step n (this can be used for music and video editing, for example). Alternatively, the independent variable may represent space, rather than time. In this case, one can use the values of the signal from points on either side of a given point in order to process the signal.

Example 2.2. The signal $y[n] = x[-n]$ is noncausal; for example, $y[-1] = x[1]$, and thus the output at negative time-steps depends on the input from positive time-steps (i.e., in the future).

The signal $y(t) = x(t)\cos(t+1)$ is causal; the $t+1$ term does not appear in the input, and thus the output at any time does not depend on values of the *input* at future times. □

Stability

The notion of **stability** is a critical system property. There are many different notions of stability that can be considered, but for the purposes of this course, we will say that a system is stable if a bounded input always leads to a bounded output. In other words, for a continuous-time system, if there exists a constant $B_1 \in \mathbb{R}_{\geq 0}$ such that the input satisfies $|x(t)| \leq B_1$ for all $t \in \mathbb{R}$, then there should exist some other constant $B_2 \in \mathbb{R}_{\geq 0}$ such that $|y(t)| \leq B_2$ for all $t \in \mathbb{R}$. An entirely analogous definition holds for discrete-time systems. Loosely speaking, for a stable system, the output cannot grow indefinitely when the input is bounded by a certain value.

Example 2.3. The system $y(t) = tx(t)$ is memoryless and causal, but not stable. For example, if $x(t) = 1$ for all $t \in \mathbb{R}$, we have $y(t) = t$ which is not bounded by any constant.

Similarly, the system $y[n] = y[n-1] + x[n]$ is not stable. This is seen by noting that $y[n] = \sum_{k=-\infty}^{n} x[k]$. So, for example, if $x[n] = u[n]$, we have $y[n] = (n+1)$ if $n \geq 0$, which is unbounded.

An example of a stable causal memoryless system is $y(t) = \cos(x(t))$. Another example of a stable and causal system is

$$y[n] = \begin{cases} 0 & \text{if } n < 0 \\ \alpha y[n-1] + x[n] & \text{if } n \geq 0 \end{cases},$$

where $\alpha \in \mathbb{R}$ satisfies $|\alpha| < 1$. Specifically, if $|x[n]| \leq B_1$ for all $n \in \mathbb{Z}$, then we have $|y[n]| \leq \frac{B_1}{1-|\alpha|}$ for all $n \in \mathbb{Z}$. To see this, we prove by induction. Clearly $|y[n]| \leq \frac{B_1}{1-|\alpha|}$ for $n \leq 0$. Suppose that $|y[n]| \leq \frac{B_1}{1-|\alpha|}$ for some $n \geq 0$. Then we have

$$\begin{aligned} |y[n+1]| = |\alpha y[n] + x[n+1]| &\leq |\alpha||y[n]| + |x[n+1]| \\ &\leq |\alpha|\frac{B_1}{1-|\alpha|} + B_1 \\ &= \frac{B_1}{1-|\alpha|}. \end{aligned}$$

Thus, by induction, we have $|y[n]| \leq \frac{B_1}{1-|\alpha|}$ for all $n \in \mathbb{Z}$. □

The above notion of stability is known as *Bounded-Input-Bounded-Output* (BIBO) stability. There are also other notions of stability, such as ensuring that the internal state of the system remains stable as well. One of the main objectives of control systems is to ensure that the overall system remains stable, as you will see in your control systems courses in later semesters.

Time-Invariance

A system is said to be **time-invariant** if the system reacts in the same way to an input signal, regardless of the time at which the input is applied. In other words, if $y(t)$ is the output signal when the input signal is $x(t)$, then the output due to $x(t-t_0)$ should be $y(t-t_0)$ for any time-shift t_0. Note that for a system to be time-invariant, this should hold for every input signal.

Example 2.4. The system $y(t) = \cos(x(t))$ is time-invariant. Suppose we define the input signal $w(t) = x(t-t_0)$ (i.e., a time-shifted version of $x(t)$). Let $y_w(t)$ be the output of the system when $w(t)$ is applied. Then we have

$$y_w(t) = \cos(w(t)) = cos(x(t-t_0)) = y(t-t_0)$$

and thus the output due to $x(t-t_0)$ is time-shifted version of $y(t)$, as required.

An example of a time-varying system is $y[n] = nx[n]$. For example, if $x[n] = \delta[n]$, then we have the output signal $y[n] = 0$ for all time. However, if $x[n] = \delta[n-1]$, then we have $y[n] = 1$ for $n = 1$ and zero for all other times. Thus a shift in the input did not result in a simple shift in the output. □

Linearity

A system is **linear** if it satisfies the following two properties.

1. **Additivity:** Suppose the output is $y_1(t)$ when the input is $x_1(t)$, and the output is $y_2(t)$ when the input is $x_2(t)$. Then the output to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$.
2. **Scaling:** Suppose the output is $y(t)$ when the input is $x(t)$. Then for any complex number α, the output should be $\alpha y(t)$ when the input is $\alpha x(t)$.

Both properties together define the *superposition property*: if the input to the system is $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, then the output should be $\alpha_1 y_1(t) + \alpha_2 y_2(t)$. Note that this must hold for any inputs and scaling parameters in order for the system to qualify as linear. An entirely analogous definition holds for discrete-time systems.

For any linear system, the output must be zero for all time when the input is zero for all time. To see this, consider any arbitrary input $x(t)$, and let the corresponding output be $y(t)$. Then, using the scaling property, the output to

$\alpha x(t)$ must be $\alpha y(t)$ for any scalar complex number α. Simply choosing $\alpha = 0$ yields the desired result that the output will be the zero signal when the input is the zero signal.

Example 2.5. The system $y(t) = tx(t)$ is linear. To see this, consider two arbitrary input signals $x_1(t)$ and $x_2(t)$, and two arbitrary scalars α_1, α_2. Then we have

$$t(\alpha_1 x_1(t) + \alpha_2 x_2(t)) = \alpha_1 t x_1(t) + \alpha_2 t x_2(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

where $y_1(t)$ and $y_2(t)$ are the outputs due to $x_1(t)$ and $x_2(t)$, respectively.

The system $y[n] = x^2[n]$ is nonlinear. Let $y_1[n] = x_1^2[n]$ and $y_2[n] = x_2^2[n]$. Consider the input $x_3[n] = x_1[n] + x_2[n]$. Then the output due to $x_3[n]$ is

$$y_3[n] = x_3^2[n] = (x_1[n] + x_2[n])^2 \neq x_1^2[n] + x_2^2[n]$$

in general. Thus the additivity property does not hold, and the system is nonlinear.

The system $y[n] = Re\{x[n]\}$ is nonlinear, where $Re\{\cdot\}$ denotes the real part of the argument. To see this, let $x[n] = a[n] + jb[n]$, where $a[n]$ and $b[n]$ are real-valued signals. Consider a scalar $\alpha = j$. Then we have

$$y[n] = Re\{x[n]\} = a[n].$$

However, $Re\{jx[n]\} = Re\{ja[n] - b[n]\} = -b[n] \neq jy[n]$. Thus, scaling the input signal by the scalar j does not result in the output being $jy[n]$, and so the scaling property does not hold.

The system $y(t) = 2x(t) + 5$ is nonlinear; it violates the fact that the all-zero input should cause the output to be zero for all time. One can also verify this by applying two different constant input signals and checking that the output due to the sum of the inputs is not equal to the sum of the corresponding outputs. □

We will be focusing almost entirely on linear time-invariant systems in this course; in practice, many systems are nonlinear and time-varying, but can often be approximated by linear time-invariant systems under certain operation conditions.

Chapter 3

Analysis of Linear Time-Invariant Systems

Reading: *Signals and Systems*, Sections 2.0-2.4.

In this part of the course, we will focus on understanding the behavior of linear time-invariant (LTI) systems. As we will see, the linearity and time-invariance properties provide a nice way to understand the input-output relationship of the system. To develop this, let us start by considering discrete-time LTI systems.

3.1 Discrete-Time LTI Systems

Consider a discrete-time system with input $x[n]$ and output $y[n]$. First, define the **impulse response** of the system to be the output when $x[n] = \delta[n]$ (i.e., the input is an impulse function). Denote this impulse response by the signal $h[n]$.

Now, consider an arbitrary signal $x[n]$. Recall from the sifting property of the impulse function that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

i.e., $x[n]$ can be written as a superposition of scaled and shifted impulse functions.

Since the system is time-invariant, the response of the system to the input $\delta[t-k]$ is $h[t-k]$. By linearity (and specifically the scaling property), the response to $x[k]\delta[n-k]$ is $x[k]\delta[n-k]$. By the additivity property, the response to $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ is then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

The above is called the **convolution sum**; the convolution of the signals $x[n]$ and $h[n]$ is denoted by

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Thus we have the following very important property of discrete-time LTI systems: **if $x[n]$ is the input signal to an LTI system, and $h[n]$ is the impulse response of the system, then the output of the system is** $y[n] = x[n] * h[n]$.

Example 3.1. Consider an LTI system with impulse response

$$h[n] = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the input signal is

$$x[n] = \begin{cases} 1 & \text{if } 0 \le n \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Since both $x[k] = 0$ for $k < 0$ and $h[n-k] = 0$ for $k > n$,

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k].$$

Thus $y[n] = 0$ for $n < 0$. When $n = 0$ we have

$$y[0] = \sum_{k=0}^{0} x[k]h[-k] = x[0]h[0] = 1.$$

When $n = 1$ we have

$$y[1] = \sum_{k=0}^{1} x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 2.$$

Similarly, $y[2] = 3$, $y[3] = 4$, $y[4] = 3$, $y[5] = 2$, $y[6] = 1$ and $y[n] = 0$ for $n \ge 7$.
□

Example 3.2. Consider an LTI system with impulse response $h[n] = u[n]$. Suppose the input signal is $x[n] = \alpha^n u[n]$ with $0 < \alpha < 1$. Then we have

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Since both $x[k] = 0$ for $k < 0$ and $h[n-k] = 0$ for $k > n$, we have

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $n \geq 0$, and $y[n] = 0$ for $n < 0$. □

3.2 Continuous-Time LTI Systems

The analysis and intuition that we developed for discrete-time LTI systems carries forward in an entirely analogous way for continuous-time LTI systems. Specifically, recall that for any signal $x(t)$, we can write

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

The expression on the right hand side is a superposition of scaled and shifted impulse functions. Thus, when this signal is applied to an LTI system, the output will be a superposition of scaled and shifted impulse responses. More specifically, if $h(t)$ is the output of the system when the input is $x(t) = \delta(t)$, then the output for a general input $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

This is the **convolution integral** and is denoted by $y(t) = x(t) * h(t)$.

Example 3.3. Suppose $x(t) = e^{-at}u(t)$ with $a \in \mathbb{R}_{>0}$ and $h(t) = u(t)$. Then the output of the LTI system with impulse response $h(t)$ is given by

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{t} e^{-a\tau}d\tau \end{aligned}$$

if $t \geq 0$, and $y(t) = 0$ otherwise. Evaluating the above expression, we have

$$y(t) = \begin{cases} \frac{1}{a}(1-e^{-at}) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

□

Example 3.4. Consider the signals

$$x(t) = \begin{cases} 1 & \text{if } 0 < t < T \\ 0 & \text{otherwise} \end{cases}, \quad h(t) = \begin{cases} t & \text{if } 0 < t < 2T \\ 0 & \text{otherwise} \end{cases},$$

where $T > 0$ is some constant. The convolution of these signals is easiest to do graphically and by considering different regions of the variable t. The result is

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 < t < T \\ Tt - \frac{1}{2}T^2 & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2 & 2T < t < 3T \\ 0 & t > 3T \end{cases}.$$

□

3.3 Properties of Linear Time-Invariant Systems

In this section we will study some useful properties of the convolution operation; based on the previous section, this will have implications for the input-output behavior of linear time-invariant systems ($h[n]$ for discrete-time systems and $h(t)$ for continuous-time systems).

3.3.1 The Commutative Property

The first useful property of convolution is that it is **commutative**:

$$x[n] * h[n] = h[n] * x[n]$$
$$x(t) * h(t) = h(t) * x(t).$$

To see this, start with the definition of convolution and perform a change of variable by setting $r = n - k$. This gives

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n].$$

The same holds for the continuous-time convolution. Thus it does not matter which of the signals we choose to flip and shift in the convolution operation.

3.3.2 The Distributive Property

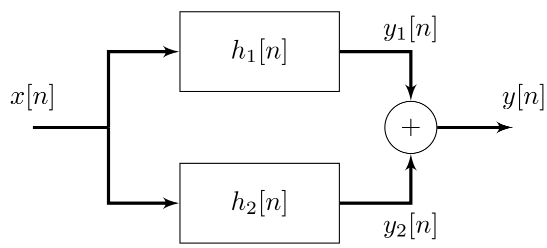
The second useful property of convolution is that it is **distributive**:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t).$$

This property is easy to verify:

$$\begin{aligned} x[n] * (h_1[n] + h_2[n]) &= \sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n]. \end{aligned}$$

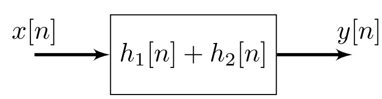
The distributive property has implications for LTI systems connected in parallel:



Let $h_1[n]$ be the impulse response of System 1, and let $h_2[n]$ be the impulse response for System 2. Then we have $y_1[n] = x[n]*h_1[n]$ and $y_2[n] = x[n]*h_2[n]$. Thus,

$$y[n] = y_1[n] + y_2[n] = x_[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n]).$$

The above expression indicates that the parallel interconnection can equivalently be viewed as $x[n]$ passing through a single system whose impulse response is $h_1[n] + h_2[n]$:



3.3.3 The Associative Property

A third useful property of convolution is that it is **associative**:

$$\begin{aligned} x[n] * (h_1[n] * h_2[n]) &= (x[n] * h_1[n]) * h_2[n] \\ x(t) * (h_1(t) * h_2(t)) &= (x(t) * h_1(t)) * h_2(t). \end{aligned}$$

In other words, it does not matter which order we do the convolutions. The above relationships can be proved by manipulating the summations (or integrals); we won't go into the details here.

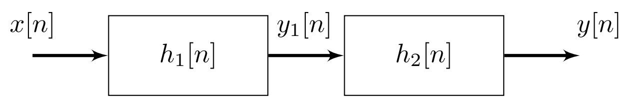


Figure 3.1: A series interconnection of systems.

Just as the distributive property had implications for parallel interconnections of systems, the associative property has implications for series interconnections of systems. Specifically, consider the series interconnection shown in Fig. 3.1.

We have

$$y[n] = y_1[n] * h_2[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]).$$

Thus, the series interconnection is equivalent to a single system with impulse response $h_1[n] * h_2[n]$, as shown in Fig. 3.2.

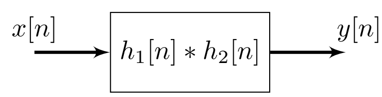


Figure 3.2: The equivalent representation of a series interconnection.

Further note that since $h_1[n] * h_2[n] = h_2[n] * h_1[n]$, we can also interchange the order of the systems in the series interconnection as shown in Fig. 3.3, without changing the overall input-output relationship between $x[n]$ and $y[n]$.

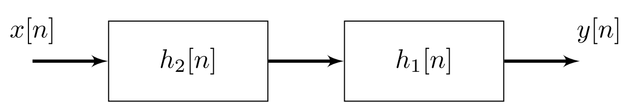


Figure 3.3: An equivalent series representation of the interconnection shown in Fig. 3.1.

3.3.4 Memoryless LTI Systems

Let us now see the implications of the memoryless property for LTI systems. Specifically, let $h[n]$ (or $h(t)$) be the impulse response of a given LTI system. Since we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k],$$

we see that $y[n]$ will depend on a value of the input signal other than at time-step n unless $h[k] = 0$ for all $k \neq 0$. In other words, for an LTI system to be memoryless, we require $h[n] = K\delta[n]$ for some constant K. Similarly, a continuous-time LTI system is memoryless if and only if $h(t) = K\delta(t)$ for some constant K. In both cases, all LTI memoryless systems have the form

$$y[n] = Kx[n] \quad \text{or} \quad y(t) = Kx(t)$$

for some constant K.

3.3.5 Invertibility of LTI Systems

Consider an LTI system with impulse response $h[n]$ (or $h(t)$). Recall that the system is said to be invertible if the output of the system uniquely specifies the input. If a system is invertible, there is another system (known as the "inverse system") that takes the output of the original system and outputs the input to the original system, as shown in Fig. 3.4.

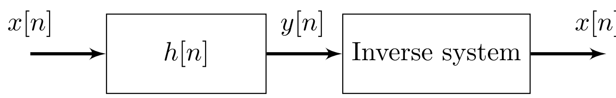


Figure 3.4: A system in series with its inverse.

Suppose the second system is LTI and has impulse response $h_I[n]$. Then, by the associative property discussed earlier, we see that the series interconnection of the system with its inverse is equivalent (from an input-output sense) to a single system with impulse response $h[n] * h_I[n]$. In particular, we require

$$x[n] = x[n] * (h[n] * h_I[n])$$

for all input signals $x[n]$, from which we have

$$h[n] * h_I[n] = \delta[n].$$

In other words, if we have an LTI system with impulse response $h[n]$, and another LTI system with impulse response $h_I[n]$ such that $h[n] * h_I[n] = \delta[n]$, then those systems are inverses of each other. The analogous statement holds in continuous-time as well.

Example 3.5. Consider the LTI system with impulse response $h[n] = \alpha^n u[n]$. One can verify that this impulse response corresponds to the system

$$y[n] = \sum_{k=-\infty}^{n} x[k]\alpha^{n-k} = \alpha y[n-1] + x[n].$$

Now consider the system $y_I[n] = x_I[n] - \alpha x_I[n-1]$, with input signal $x_I[n]$ and output signal $y_I[n]$. The impulse response of this system is $h_I[n] = \delta[n] -$

$\alpha\delta[n-1]$. We have

$$\begin{aligned} h[n] * h_I[n] &= \alpha^n u[n] * (\delta[n] - \alpha\delta[n-1]) \\ &= \alpha^n u[n] * \delta[n] - (\alpha^n u[n]) * (\alpha\delta[n-1]) \\ &= \alpha^n u[n] - \alpha(\alpha^{n-1} u[n-1]) \\ &= \alpha^n (u[n] - u[n-1]) \\ &= \alpha^n \delta[n] \\ &= \delta[n]. \end{aligned}$$

Thus, the system with impulse response $h_I[n]$ is the inverse of the system with impulse response $h[n]$. □

3.3.6 Causality of LTI Systems

Recall that a system is causal if its output at time t depends only on the inputs up to (and potentially including) t. To see what this means for LTI systems, consider the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k],$$

where the second expression follows from the commutative property of the convolution. In order for $y[n]$ to not depend on $x[n+1], x[n+2], \ldots$, we see that $h[k]$ must be zero for $k < 0$. The same conclusion holds for continuous-time systems, and thus we have the following: **A continuous-time LTI system is causal if and only if its impulse response $h(t)$ is zero for all $t < 0$. A discrete-time LTI system is causal if and only if its impulse response $h[n]$ is zero for all $n < 0$.**

Note that causality is a property of a system; however we will sometimes refer to a signal as being causal, by which we simply mean that its value is zero for n or t less than zero.

3.3.7 Stability of LTI Systems

To see what the LTI property means for stability of systems, consider again the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Note that

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |x[k]h[n-k]|$$
$$= \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]|.$$

Now suppose that $x[n]$ is bounded, i.e., there exists some $B \in \mathbb{R}_{\geq 0}$ such that $|x[n]| \leq B$ for all $n \in \mathbb{Z}$. Then the above expression becomes

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[n-k]|.$$

Thus, if $\sum_{k=-\infty}^{\infty} |h[n-k]| < \infty$ (which means that $h[n]$ is **absolutely summable**), then $|y[n]|$ will also be bounded for all n. It turns out that this is a necessary condition as well: if $\sum_{k=-\infty}^{\infty} |h[n-k]| = \infty$, then there is a bounded input that causes the output to be unbounded.

The same conclusion holds in continuous-time as well. Thus, we have: **A continuous-time LTI system is stable if and only if $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$. A discrete-time LTI system is stable if and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.**

Example 3.6. Consider the LTI system with impulse response $h[n] = \alpha^n u[n]$, where $\alpha \in \mathbb{R}$. We have

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |\alpha|^k = \begin{cases} \frac{1}{1-|\alpha|} & \text{if } |\alpha| < 1 \\ \infty & \text{if } |\alpha| \geq 1 \end{cases}.$$

Thus, the system is stable if and only if $|\alpha| < 1$.

Similarly, consider the continuous-time LTI system with impulse response $h(t) = e^{\alpha t}u(t)$, where $\alpha \in \mathbb{R}$. We have

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau = \int_{0}^{\infty} e^{\alpha\tau} d\tau = \frac{1}{\alpha}(e^{\alpha\tau})\Big|_0^{\infty} = \begin{cases} -\frac{1}{\alpha} & \text{if } \alpha < 0 \\ \infty & \text{if } \alpha \geq 0 \end{cases}.$$

Thus, the system is stable if and only if $\alpha < 0$. □

3.3.8 Step Response of LTI Systems

Just as we defined the impulse response of a system to be the output of the system when the input is an impulse function, we define the **step response** of a system to be the output when the input is a step function $u[n]$ (or $u(t)$ in continuous-time). We denote the step response as $s[n]$ for discrete-time systems and $s(t)$ for continuous-time systems.

To see how the step response is related to the impulse response, note that

$$s[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=-\infty}^{\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k].$$

This is equivalent to $s[n] = s[n-1] + h[n]$. Thus, the step response of a discrete-time LTI system is the running sum of the impulse response.

Note that this could also have been seen by noting that $\delta[n] = u[n] - u[n-1]$. If the impulse is applied to an LTI system, we get the impulse response $h[n]$. However, by the linearity property, this output must be the superposition of the outputs due to $u[n]$ and $u[n-1]$. By the time-invariance property, the output due to $u[n-1]$ is $s[n-1]$, and thus for LTI systems we have $h[n] = s[n] - s[n-1]$, which corroborates what we obtained above.

For continuous-time systems, we have the same idea:

$$s(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau.$$

Differentiating both sides and applying the fundamental theorem of calculus, we have

$$\frac{ds}{dt} = h(t),$$

i.e., the impulse response is the derivative of the step response.

3.4 Differential and Difference Equation Models for Causal LTI Systems

As we have already seen in a few examples, many systems can be described using differential equation (in continuous-time) or difference-equation (in discrete-time) models, capturing the relationship between the input and the output. For example, for a vehicle with velocity $v(t)$ and input acceleration $a(t)$, we have

$$\frac{dv}{dt} = a(t).$$

If we included wind resistance or friction (which produces a force that is proportional to the velocity in the opposite direction of travel), we have

$$\frac{dv}{dt} = -\alpha v(t) + a(t),$$

where $\alpha > 0$ is the coefficient of friction. Similarly, given an RC circuit, if we define the voltage across the capacitor as the output, and the source voltage as the input, then the input and output are again related via a differential equation of the above form.

In discrete-time, consider a bank-account where earnings are deposited at the end of each month. Let the amount in the account at the end of month n be denoted by $s[n]$. Then we have

$$s[n] = (1 + r)s[n - 1] + x[n]$$

where r is the interest rate and $x[n]$ is the new amount deposited into the account at the end of month n.

Since such differential and difference equations play a fundamental role in the analysis of LTI systems, we will now review some methods to solve such equations.

3.4.1 Linear Constant-Coefficient Differential Equations

To illustrate the solution of linear differential equations, we consider the following example.

Example 3.7. Consider the differential equation

$$\frac{dy}{dt} + 2y(t) = x(t),$$

where $x(t) = Ke^{3t}u(t)$ (K is some constant). The solution to such differential equations is given by $y(t) = y_h(t) + y_p(t)$, where $y_p(t)$ is a **particular solution** to the above equation, and $y_h(t)$ is a *homogeneous solution* satisfying the differential equation

$$\frac{dy_h}{dt} + 2y_h(t) = 0.$$

The above differential equation is called **homogeneous** as it has no driving function $x(t)$.

Let us first solve the homogeneous equation. For equations of this form (where a sum of derivatives of $y_h(t)$ have to sum to zero), a reasonable guess would be that $y_h(t)$ takes the form

$$y_h(t) = Ae^{mt}$$

for some $m \in \mathbb{C}$. Substituting this into the homogeneous equation gives

$$mAe^{mt} + 2Ae^{mt} = 0 \Rightarrow m + 2 = 0 \Rightarrow m = -2.$$

Thus, the homogeneous solution is $y_h(t) = Ae^{-2t}$, for any constant A.

Next, we search for a particular solution to the equation

$$\frac{dy_p}{dt} + 2y_p(t) = Ke^{3t}u(t).$$

It seems reasonable to try $y_p(t) = Be^{3t}$, for some constant B. Substituting and evaluating for $t > 0$, we have

$$3Be^{3t} + 2Be^{3t} = Ke^{3t} \Rightarrow B = \frac{K}{5}.$$

Thus, the particular solution is given by $y_p(t) = \frac{K}{5}e^{3t}$ for $t > 0$.

Together, we have $y(t) = y_h(t) + y_p(t) = Ae^{-2t} + \frac{K}{5}e^{3t}$ for $t > 0$. Note that the coefficient A has not been determined yet; in order to do so, we need more information about the solutions to the differential equation, typically in the form of initial conditions. For example, if we know that the system is *at rest* until the input is applied (i.e., $y(t) = 0$ until $x(t)$ becomes nonzero), we have $y(t) = 0$ for $t < 0$. Suppose we are given the initial condition $y(0) = 0$. Then,

$$y(0) = A + \frac{K}{5} = 0 \Rightarrow A = -\frac{K}{5}.$$

Thus, with the given initial condition, we have $y(t) = \frac{K}{5}(e^{3t} - e^{-2t})u(t)$. □

The above example illustrates the general approach to solving linear differential equations of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}.$$

First find the homogeneous solution to the equation

$$\sum_{k=0}^{N} a_k \frac{d^k y_h}{dt^k} = 0$$

by hypothesizing that $y_h(t) = Ae^{mt}$ for some $m \in \mathbb{C}$. If there are N different values of m, denoted $m_1, m_2, \ldots, m_N$ for which the proposed form holds, then we take the homogeneous solution to be $y_h(t) = A_1e^{m_1t} + A_2e^{m_2t} + \cdots + A_Ne^{m_Nt}$, where the coefficients $A_1, \ldots, A_N$ are to be determined from initial conditions. If there are not N different values of m, then further work is required; we will see a more general way to solve these cases later in the course.

Next, find a particular solution to the equation

$$\sum_{k=0}^{N} a_k \frac{d^k y_p}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x}{dt^k}$$

where $x(t)$ is some given function. The idea will be to make $y_p(t)$ a linear combination of terms that, when differentiated, yield terms that appear in $x(t)$ and its derivatives. Typically this only works when $x(t)$ involves terms like $e^t, \sin(t), \cos(t)$, polynomials in t, etc. Let's try another example.

Example 3.8. Consider the differential equation

$$y''(t) + y'(t) - 6y(t) = x'(t) + x(t), \tag{3.1}$$

where $x(t) = e^{4t}u(t)$.

We first search for a homogeneous solution $y_h(t) = Ae^{mt}$ satisfying

$$y_h''(t) + y_h'(t) - 6y_h(t) = 0 \Rightarrow m^2Ae^{mt} + mAe^{mt} - 6Ae^{mt} = 0 \Rightarrow (m^2 + m - 6) = 0.$$

This yields $m = -3$ or $m = 2$. Thus, the homogeneous solution is of the form

$$y_h(t) = A_1 e^{-3t} + A_2 e^{2t}$$

for some constants A_1 and A_2 that will be determined from the initial conditions.

To find a particular solution, note that for $t > 0$, we have $x'(t) + x(t) = 4e^{4t} + e^{4t} = 5e^{4t}$. Thus we search for a particular solution of the form $y_p(t) = Be^{4t}$ for $t > 0$. Substituting into the differential equation (3.1), we have

$$y_p''(t) + y_p'(t) - 6y_p(t) = x'(t) + x(t) \Rightarrow 16Be^{4t} + 4Be^{4t} - 6Be^{4t} = 5e^{4t} \Rightarrow B = \frac{5}{14}.$$

Thus, $y_p(t) = \frac{5}{14}e^{4t}$ for $t > 0$ is a particular solution.

The overall solution is then of the form $y(t) = y_h(t) + y_p(t) = A_1 e^{-3t} + A_2 e^{2t} + \frac{5}{14}e^{4t}$ for $t > 0$. If we are told that the system is at rest until the input is applied, and that $y(0) = y'(0) = 0$, we have

$$\begin{aligned} y(0) &= A_1 + A_2 + \frac{5}{14} = 0 \\ y'(0) &= -3A_1 + 2A_2 + \frac{20}{14} = 0. \end{aligned}$$

Solving these equations, we obtain $A_1 = \frac{1}{7}$ and $A_2 = -\frac{1}{2}$. Thus, the solution is

$$y(t) = \left(\frac{1}{7}e^{-3t} - \frac{1}{2}e^{2t} + \frac{5}{14}e^{4t}\right) u(t).$$

□

3.4.2 Linear Constant Coefficient Difference Equations

The same general idea that we used to solve differential equations in the previous section apply to solving difference equations of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{3.2}$$

The overall solution will be of the form $y[n] = y_h[n] + y_p[n]$, where $y_h[n]$ is a homogeneous solution to

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0,$$

and $y_p[n]$ is a particular solution satisfying the difference equation (3.2) for the given function $x[n]$. In this case, we seek homogeneous solutions of the form $y_h[n] = A\beta^n$ for some $A, \beta \in \mathbb{C}$, and seek particular solutions that have the same form as the quantities that appear in $x[n]$. Let's do an example.

Example 3.9. Suppose we have the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \tag{3.3}$$

with $x[n] = \left(\frac{1}{3}\right)^n u[n]$.

The solution to this difference equation will be of the form $y[n] = y_h[n] + y_p[n]$, where $y_h[n]$ is the homogeneous solution satisfying

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0$$

and $y_p[n]$ is a particular solution to the given difference equation with the given input signal $x[n]$.

To find the homogeneous solution, we try $y_h[n] = A\beta^n$ for some constants A and β. Substituting into the homogeneous difference equation, we obtain

$$y_h[n] - \frac{1}{2}y_h[n-1] = A\beta^n - \frac{A}{2}\beta^{n-1} = 0 \Rightarrow \beta = \frac{1}{2}.$$

Thus, the homogeneous solution is $y_h[n] = A\beta^n$ for some A that we will identify based on initial conditions (after we have found the particular solution).

To find the particular solution, we attempt to mimic the input. Thus, we seek a particular solution of the form $y_p[n] = B\left(\frac{1}{3}\right)^n$ for $n \geq 0$ and some constant B. Substituting this into the difference equation (3.3), we have

$$B\left(\frac{1}{3}\right)^n - \frac{B}{2}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{3}^n$$

for $n \geq 1$ (note that we don't look at $n = 0$ here because we have not defined $y_p[-1]$). Solving this, we get $B = -2$. Thus the particular solution is $y_p[n] = -2\left(\frac{1}{3}\right)^n$ for $n \geq 0$.

Now, we have $y[n] = y_h[n] + y_p[n] = A\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n$ for $n \geq 0$. Suppose we are told that the system is at rest for $n < 0$, i.e., $y[n] = 0$ for $n < 0$. Looking at equation (3.3), we have

$$y[0] - \frac{1}{2}y[-1] = 1 \Rightarrow y[0] = 1.$$

Substituting the expression for $y[n]$, we have

$$1 = y[0] = A - 2 \Rightarrow A = 3.$$

Thus, the solution is given by

$$y[n] = \left(3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right)u[n].$$

□

An alternative method to solve difference equations is to write them in **recursive form**, and then iteratively solve, as shown by the following example.

Example 3.10. Suppose

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

We can rewrite this as

$$y[n] = \frac{1}{2}y[n-1] + x[n].$$

Suppose that $x[n] = \delta[n]$ and the system is initially at rest (i.e., $y[n] = 0$ for $n < 0$). Then we have

$$\begin{aligned} y[0] &= \frac{1}{2}y[-1] + \delta[0] = 1 \\ y[1] &= \frac{1}{2}y[0] + \delta[1] = \frac{1}{2} \\ y[2] &= \frac{1}{2}y[1] + \delta[2] = \frac{1}{4} \\ &\vdots \\ y[n] &= \left(\frac{1}{2}\right)^n \end{aligned}$$

Thus, the impulse response is $h[n] = \left(\frac{1}{2}\right)^n u[n]$.[1] □

3.5 Block Diagram Representations of Linear Differential and Difference Equations

It is often useful to represent linear differential and difference equations using block diagrams; this provides us with a way to implement such equations using primitive computational elements (form form the components of the block diagram), and to derive alternative representations of systems. Here, we will focus on differential and difference equations of the form

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_0 y(t) = b_0 x(t)$$
$$y[n+N] + a_{N-1}y[n+N-1] + \cdots + a_0 y[n] = b_0 x[n].$$

Drawing block diagrams for more general differential and difference equations (involving more than just $x[n]$ on the right hand side) is easier using Laplace

[1] One can also calculate this using the homogeneous and particular solutions; in this case, the particular solution would have the form $y_p[n] = B\delta[n]$ and B would be found to be zero, so that $y[n] = y_h[n] = A\left(\frac{1}{2}\right)^n$. Under the condition of initial rest and $y[0] = 1$ (obtained from the difference equation), we obtain $A = 1$, thus matching the impulse response calculated recursively above.

and z-transform techniques, and so we will defer a study of such equations until then.

For the above equations, we start by writing the highest derivative of y (or the most advanced version of y) in terms of all of the other quantities:

$$\frac{d^N y(t)}{dt^N} = -a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} - \cdots - a_0 y(t) + b_0 x(t) \tag{3.4}$$

$$y[n+N] = -a_{N-1}y[n+N-1] - \cdots - a_0 y[n] + b_0 x[n]. \tag{3.5}$$

Next, we use a key building block: the **integrator block** (for continuous-time) or the **delay block** (for discrete-time). Specifically, the integrator block is a system whose output is the integral of the input, and the delay block is a system whose output is a delayed version of the input. Thus, if we feed $\frac{dy}{dt}$ into the integrator block, we get $y(t)$ out, and if we feed $y[n+1]$ into the delay block, we get $y[n]$ out, as shown in Fig. 3.5.

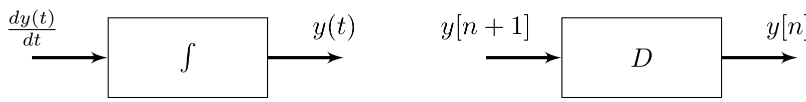


Figure 3.5: Integrator and Delay blocks.

To use these blocks to represent differential and difference equations, we simply chain a sequence of these blocks in series, and feed the highest derivative into the first block in the chain, as shown in Fig. 3.6.

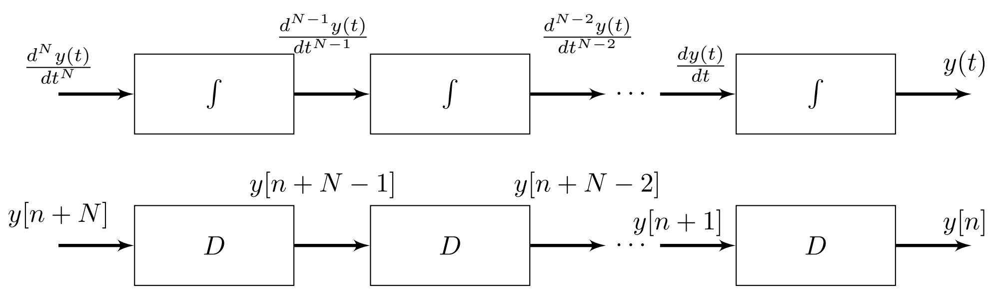


Figure 3.6: Chained integrator and delay blocks.

This series chain of integrator (or delay) blocks provides us with all of the signals needed to represent (3.4) and (3.5). Specifically, from equation (3.4), we see that $\frac{d^N y(t)}{dt^N}$ is a linear combination of the signals $\frac{d^{N-1}y(t)}{dt^{N-1}}, \cdots, y(t), x(t)$. Thus, to generate the signal $\frac{d^N y(t)}{dt^N}$, we simply take the signals from the corresponding integrator blocks, multiply them by the coefficients, and add them all together. The same holds true for the signal $y[n+N]$ in (3.5).

Chapter 4

Fourier Series Representation of Periodic Signals

Reading: *Signals and Systems*, Chapter 3.

In the last part of the course, we decomposed signals into sums of scaled and time-shifted impulse functions. For LTI systems, we could then write the output as a sum of scaled and time-shifted impulse responses (using the superposition property). In this part of the course, we will consider alternate (and very useful) decompositions of signals as sums of scaled **complex exponential** functions. As we will see, such functions exhibit some nice behavior when applied to LTI systems. This particular chapter will focus on decomposing periodic signals into complex exponentials (leading to the **Fourier Series**), and subsequent chapters will deal with the decomposition of more general signals.

4.1 Applying Complex Exponentials to LTI Systems

Recall that a complex exponential has the form $x(t) = e^{st}$ (in continuous-time), and $x[n] = z^n$ (in discrete-time), where s and z are general complex numbers. Let's start with a continuous-time LTI system with impulse response $h(t)$. When we apply the complex exponential $x(t) = e^{st}$ to the system, the output is given by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau$$
$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

Let us define

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau.$$

If, for the given complex number s, the above integral exists (i.e., is finite), then $H(s)$ is just some complex number. Thus, we see that for an LTI system, if we apply the complex exponential $x(t) = e^{st}$ as an input, we obtain the quantity

$$y(t) = H(s)e^{st}$$

as an output. In other words, we get the *same complex exponential* out of the system, just scaled by the complex number $H(s)$. Thus, the signal e^{st} is called an **eigenfunction** of the system, with **eigenvalue** $H(s)$.

The same reasoning applies for discrete-time LTI systems. Consider an LTI system with impulse response $h[n]$, and input $x[n] = z^n$. Then,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$
$$= z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

Let us define

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}.$$

If this sum converges for the given choice of complex number z, then $H(z)$ is just some complex number. Thus, we see again that for a discrete-time LTI system with the complex exponential $x[n] = z^n$ as an input, we obtain the quantity

$$y[n] = H(z)z^n$$

as an output. In this case z^n is an eigenfunction of the system, and $H(z)$ is the eigenvalue.

So, to summarize, we have the following:

- If the signal $x(t) = e^{st}$ is applied to an LTI system with impulse response $h(t)$, the output is $y(t) = H(s)e^{st}$, where $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$ (assuming the integral exists).
- If the signal $x[n] = z^n$ is applied to an LTI system with impulse response $h[n]$, the output is $y[n] = H(z)z^n$, where $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ (assuming the sum converges).

As we will see later in the course, the quantities $H(s)$ and $H(z)$ are the **Laplace Transform** and **z-Transform** of the impulse response of the system, respectively.

Note that the above translates to superpositions of complex exponentials in an natural way. Specifically, if the input is $x(t) = \sum_{i=1}^{n} a_i e^{s_i t}$ for some complex numbers $a_1, \ldots, a_n$ and $s_1, \ldots, s_n$, we have

$$y(t) = \sum_{i=1}^{n} a_i H(s_i) e^{s_i t}.$$

An essentially identical relationship is true for discrete-time systems.

Example 4.1. Consider the signal $x(t) = \cos(\omega t)$. We can write this as

$$x(t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}.$$

Thus, the output will be

$$y(t) = \frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)e^{-j\omega t}.$$

Now, suppose that the impulse response of the system is real-valued (i.e., $h(t) \in \mathbb{R}$ for all t). Then, we have

$$H(-j\omega)^* = \int_{-\infty}^{\infty} \left(h(\tau)e^{j\omega\tau}\right)^* d\tau = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(j\omega).$$

Thus, for real-valued impulse responses, we have $H(-j\omega) = H(j\omega)^*$. We can equivalently write these in polar form as

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}, \quad H(-j\omega) = |H(j\omega)|e^{-j\angle H(j\omega)}.$$

Thus,

$$\begin{aligned} y(t) &= \frac{1}{2}H(j\omega)e^{j\omega t} + \frac{1}{2}H(-j\omega)e^{-j\omega t} \\ &= \frac{1}{2}|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t} + \frac{1}{2}|H(j\omega)|e^{-j\angle H(j\omega)}e^{-j\omega t} \\ &= |H(j\omega)|\cos(\omega t + \angle H(j\omega)). \end{aligned}$$

□

Example 4.2. Consider the system $y(t) = x(t-t_0)$, where $t_0 \in \mathbb{R}$. The impulse response if this system is $h(t) = \delta(t - t_0)$, and thus

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-st}dt = e^{-st_0}.$$

Suppose we apply the signal $x(t) = \cos(\omega_0 t)$ to the system. We expect the output to be $\cos(\omega_0(t - t_0))$, based on the definition of the system. Let's verify this using the identities we derived earlier. We have

$$|H(j\omega_0)| = |e^{-j\omega_0 t_0}| = 1, \quad \angle H(j\omega_0) = -\omega_0 t_0.$$

Thus, when we apply the input $\cos(\omega_0 t)$, the output is given by

$$y(t) = |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0)) = \cos(\omega_0 t - \omega_0 t_0),$$

matching what we expect. □

4.2 Fourier Series Representation of Continuous-Time Periodic Signals

Consider the complex exponential signal

$$x(t) = e^{j\omega_0 t}.$$

Recall that this signal is periodic with fundamental period $T = \frac{2\pi}{\omega_0}$ (assuming $\omega_0 > 0$). Based on this complex exponential, we can define an entire *harmonic family* of complex exponentials, given by

$$\phi_k(t) = e^{jk\omega_0 t}, k \in \mathbb{Z}.$$

In other words, for each $k \in \mathbb{Z}$, $\phi_k(t)$ is a complex exponential whose fundamental frequency is $k\omega_0$ (i.e., k times the fundamental frequency of $x(t)$). Thus, each of the signals $\phi_k(t)$ is periodic with period T, since

$$\phi_k(t + T) = e^{jk\omega_0(t+T)} = e^{jk\omega_0 t}e^{jk\omega_0 T} = e^{jk\omega_0 t}e^{jk2\pi} = e^{jk\omega_0 t}.$$

Note that T may not be the *fundamental period* of the signal $\phi_k(t)$, however.

Since each of the signals in the harmonic family is periodic with period T, a linear combination of signals from that family is also periodic. Specifically, consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}.$$

The terms corresponding to $k = 1$ and $k = -1$ are known as the *first harmonic* of the signal $x(t)$. The terms corresponding to $k = 2$ and $k = -2$ are known as the *second harmonic* and so forth.

Suppose we are given a certain periodic signal $x(t)$ with fundamental period T. Define $\omega_0 = \frac{2\pi}{T}$ and suppose that we can write $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \quad \left(= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right)$$

for some sequence of coefficients a_k, $k \in \mathbb{Z}$. Then the above representation is called the **Fourier Series** representation of $x(t)$. The quantities a_k, $k \in \mathbb{Z}$ are called the **Fourier Series coefficients**.

Example 4.3. Consider the signal $x(t) = \cos(\omega_0 t)$, where $\omega_0 > 0$. We have

$$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}.$$

This is the Fourier Series representation of $x(t)$; it has only first harmonics, with coefficients $a_1 = a_{-1} = \frac{1}{2}$.

Similarly, consider the signal $x(t) = \sin(\omega_0 t)$. We have

$$x(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}.$$

Once again, the signal has only first harmonics, with coefficients $a_1 = \frac{1}{2j}$ and $a_{-1} = -\frac{1}{2j}$. □

Suppose that we have a periodic signal that has a Fourier Series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{4.1}$$

Now suppose that $x(t)$ is real, i.e., $x^*(t) = x(t)$. Taking the complex conjugate of both sides of the above expression, we have

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}.$$

Equating the expressions for $x(t)$ and $x^*(t)$, we have

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}.$$

Comparing the terms, we see that for any $k \in \mathbb{Z}$, the coefficient of $e^{jk\omega_0 t}$ is a_k on the left hand side, and is a_{-k}^* on the right hand side. Thus, for real signals $x(t)$, the Fourier Series coefficients satisfy

$$a_{-k} = a_k^*$$

for all $k \in \mathbb{Z}$. Substituting this into the Fourier Series representation (4.1) we have

$$\begin{aligned} x(t) &= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right) \\ &= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right) \\ &= a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\}, \end{aligned}$$

where Re is the real part of the given complex number. If we write a_k in polar form as $r_k e^{j\theta_k}$, the above expression becomes

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left\{r_k e^{j(k\omega_0 t + \theta_k)}\right\} = a_0 + 2\sum_{k=1}^{\infty} r_k \cos(k\omega_0 t + \theta_k).$$

This is an alternate representation of the Fourier Series for real-valued signals (known as the trigonometric representation).

4.3 Calculating the Fourier Series Coefficients

Suppose that we are given a periodic signal $x(t)$ with period T, and that this signal has a Fourier Series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

We will soon see conditions under which a signal will have such a representation, but for now, suppose that we are just interested in finding the coefficients a_k, $k \in \mathbb{Z}$. To do this, multiply $x(t)$ by $e^{-jn\omega_0 t}$, where n is some integer. This gives

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}.$$

Now suppose that we integrate both sides of the above equation from t_0 to t_0+T for any t_0:

$$\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t} dt = \int_{t_0}^{t_0+T} \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \int_{t_0}^{t_0+T} e^{j(k-n)\omega_0 t} dt.$$

Now note that if $n = k$, we have

$$\int_{t_0}^{t_0+T} e^{j(k-n)\omega_0 t} dt = \int_{t_0}^{t_0+T} 1 dt = T.$$

Otherwise, if $n \neq k$, we have

$$\begin{aligned}\int_{t_0}^{t_0+T} e^{j(k-n)\omega_0 t}dt &= \frac{1}{j(k-n)\omega_0}\ e^{j(k-n)\omega_0 t}\Big|_{t_0}^{t_0+T}\\ &= \frac{1}{j(k-n)\omega_0}\left(e^{j(k-n)\omega_0(t_0+T)} - e^{j(k-n)\omega_0 t_0}\right)\\ &= 0.\end{aligned}$$

Thus, we have

$$\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_{t_0}^{t_0+T} e^{j(k-n)\omega_0 t}dt = a_n T,$$

or equivalently,

$$a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt$$

where t_0 is any arbitrary starting point. In other words, we obtain the Fourier coefficient a_n by multiplying the signal $x(t)$ by $e^{-jn\omega_0 t}$ and then integrating the resulting product over any period.

Example 4.4. Consider the signal

$$x(t) = \begin{cases} 0 & -\frac{T}{2} \leq t < T_1 \\ 1 & -T_1 \leq t \leq T_1 \\ 0 & T_1 < t < \frac{T}{2} \end{cases},$$

where $T_1 \leq T$ and $x(t)$ is T-periodic.

Define $\omega_0 = \frac{2\pi}{T}$. We have

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{T_1}^{T_1} x(t)dt = \frac{2T_1}{T}.$$

For $k \neq 0$, we have

$$\begin{aligned}a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt &= \frac{1}{T}\frac{1}{-jk\omega_0}\ e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}\\ &= \frac{2}{Tk\omega_0}\sin(k\omega_0 T_1)\\ &= \frac{\sin(k\omega_0 T_1)}{k\pi},\end{aligned}$$

where we used the fact that $T\omega_0 = 2\pi$.

Note that in this case, a_k is real for all $k \in \mathbb{Z}$, and satisfies $a_k = a_{-k}$. Thus we can also write the Fourier series as

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) = a_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\omega_0 t).$$

□

4.3.1 A Vector Analogy for the Fourier Series

In the derivation of the Fourier series coefficients, we saw that

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t}e^{-jn\omega_0 t}dt = 0$$

if $k \neq n$, and is equal to T otherwise. The functions $e^{jk\omega_0}$ and $e^{jn\omega_0}$ (for $k \neq n$) are said to be **orthogonal**. More generally, a set of functions $\phi_k(t)$, $k \in \mathbb{Z}$, are said to be orthogonal on an interval $[a, b]$ if

$$\int_a^b \phi_k(t)\phi_n^*(t) = 0$$

if $k \neq n$, and nonzero otherwise. Note that $\phi_n^*(t)$ is the complex conjugate of $\phi_n(t)$. We then derived the expressions for the coefficients by using the orthogonality property. However, that derivation assumed that the signal could be written as a linear combination of the functions in the harmonic family, and then derived the coefficient expressions. Here will justify this by first trying to *approximate* a given signal by a finite number of functions from the harmonic family and then taking the number of approximating functions to infinity. We will start by reviewing how to approximate a given vector by other vectors, and then explore the analogy to the approximation of functions.

Review of Approximation of Vectors

The above definition of orthogonality of functions is exactly analogous to the definition of orthogonal vectors in a vector space. Recall that two vectors $v_1, v_2 \in \mathbb{R}^n$ are said to be orthogonal if $v_1'v_2 = 0$. Suppose we are given the vectors

$$x = \begin{bmatrix}1\\2\\3\end{bmatrix},\quad v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Note that $v_1'v_2 = 0$ and thus v_1 and v_2 are orthogonal. Suppose we wish to approximate the vector x using a linear combination of the vectors v_1 and v_2. In other words, we wish to find coefficients a and b so that the approximation

$$\hat{x} = av_1 + bv_2$$

is "close" to x. A typical metric of "closeness" is taken to be the square of the approximation error. Specifically, the approximation error is given by

$$e = x - \hat{x} = x - av_1 - bv_2.$$

Note that e is a vector, where the i-th component is the approximation error for the i-th component of x. We try to minimize $e_1^2 + e_2^2 + e_3^2$, which is given by

$$\begin{aligned} e_1^2 + e_2^2 + e_3^2 = e'e &= (x - av_1 - bv_2)'(x - av_1 - bv_2) \\ &= x'x - ax'v_1 - bx'v_2 - av_1'x + a^2 v_1'v_1 + abv_1'v_2 \\ &\quad - bv_2'x + abv_2'v_1 + b^2 v_2^2. \end{aligned}$$

Noting that v_1 and v_2 are orthogonal, we have

$$e'e = x'x - ax'v_1 - bx'v_2 - av_1'x + a^2 v_1'v_1 - bv_2'x + b^2 v_2^2.$$

This is a convex function of the scalars a and b. If we wish to minimize $e'e$, we take the derivative with respect to these scalars and set it equal to zero. This yields

$$\begin{aligned} \frac{\partial e'e}{\partial a} &= -x'v_1 - v_1'x + 2av_1'v_1 = 0 \\ \Rightarrow a &= \frac{v_1'x}{v_1'v_1} \\ \frac{\partial e'e}{\partial b} &= -x'v_2 - v_2'x + 2bv_2'v_2 = 0 \\ \Rightarrow b &= \frac{v_2'x}{v_2'v_2}, \end{aligned}$$

where we used the fact that $x'v_1 = v_1'x$ and $x'v_2 = v_2'x$ (since these quantities are all scalars). In terms of the vectors given above, we obtain

$$a = \frac{1}{1} = 1, \quad b = \frac{5}{2}.$$

Application to Approximation of Functions

Entirely analogous ideas hold when we are trying to approximate one function as a linear combination of a set of orthogonal functions (as in the Fourier series). Given a set of orthogonal functions $\phi_k(t)$ over an interval $[a, b]$, suppose we wish to approximate a given function $x(t)$ as a linear combination of some finite number of these functions. Specifically, suppose that we are given some positive integer N, and wish to find the best coefficients $\alpha_k \in \mathbb{C}$ such that the estimate

$$\hat{x}(t) = \sum_{k=-N}^{N} a_k \phi_k(t)$$

is "close" to $x(t)$. Mathematically, we will use the notion of **squared error** to measure "closeness." Specifically, the approximation error at any given point in time t is given by

$$e(t) = x(t) - \hat{x}(t) = x(t) - \sum_{k=-N}^{N} a_k \phi_k(t),$$

and the squared error over the entire interval $[a, b]$ is then defined as

$$\int_a^b |e(t)|^2 dt.$$

Here, we will allow $e(t)$ to a general complex valued function, so the absolute value in the integral is interpreted as the magnitude of the complex number $e(t)$, i.e., the square error over the interval $[a, b]$ is given by

$$\int_a^b e^*(t)e(t)dt.$$

Consider the harmonic family $\phi_k(t) = e^{jk\omega_0 t}$, and suppose that we wish to find the best approximation of a given T-periodic signal $x(t)$ as a linear combination of $\phi_k(t)$ for $-N \leq k \leq N$, i.e.,

$$\hat{x}(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t},$$

with error

$$e(t) = x(t) - \hat{x}(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

We evaluate the squared error over any interval of length T (since the functions $\phi_k(t)$ are orthogonal over such intervals):

$$\begin{aligned}
\text{Squared Error} &= \int_{t_0}^{t_0+T} |e(t)|^2 dt = \int_{t_0}^{t_0+T} e^*(t)e(t)dt \\
&= \int_{t_0}^{t_0+T} \left(x^*(t) - \sum_{k=-N}^{N} a_k^* e^{-jk\omega_0 t} \right) \left(x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \right) dt \\
&= \int_{t_0}^{t_0+T} \left(x^*(t)x(t) - x^*(t) \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} - x(t) \sum_{k=-N}^{N} a_k^* e^{-jk\omega_0 t} \right) dt \\
&\quad + \int_{t_0}^{t_0+T} \left(\sum_{k=-N}^{N} a_k^* e^{-jk\omega_0 t} \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \right) dt \\
&= \int_{t_0}^{t_0+T} |x(t)|^2 dt + \int_{t_0}^{t_0+T} \left(-x^*(t) \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} - x(t) \sum_{k=-N}^{N} a_k^* e^{-jk\omega_0 t} \right) dt \\
&\quad + \int_{t_0}^{t_0+T} \left(\sum_{k=-N}^{N} \sum_{k=-N}^{N} a_k^* a_n e^{-jk\omega_0 t} e^{jn\omega_0 t} \right) dt
\end{aligned}$$

$$= \int_{t_0}^{t_0+T} |x(t)|^2 dt - \sum_{k=-N}^{N} a_k \int_{t_0}^{t_0+T} x^*(t) e^{jk\omega_0 t} dt - \sum_{k=-N}^{N} a_k^* \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$
$$+ \left(\sum_{k=-N}^{N} \sum_{n=-N}^{N} a_k^* a_n \int_{t_0}^{t_0+T} e^{-jk\omega_0 t} e^{jn\omega_0 t} \right) dt$$
$$= \int_{t_0}^{t_0+T} |x(t)|^2 dt - \sum_{k=-N}^{N} a_k \int_{t_0}^{t_0+T} x^*(t) e^{jk\omega_0 t} dt - \sum_{k=-N}^{N} a_k^* \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$$
$$+ T \sum_{k=-N}^{N} |a_k|^2,$$

where we used the fact that $\int_{t_0}^{t_0+T} e^{-jk\omega_0 t} e^{jn\omega_0 t} dt = 0$ if $k \neq n$ and T otherwise.

Our job is to find the best coefficients $a_k, -N \le k \le N$ to minimize the square error. Thus, we first write $a_k = b_k + jc_k$, where $b_k, c_k \in \mathbb{R}$, and then differentiate the above expression with respect to b_k and c_k and set the result to zero. After some algebra, we obtain the optimal coefficient as

$$a_k = b_k + jc_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt,$$

which is exactly the same expression we found for the Fourier series coefficients earlier. Note again why we bothered to go through this exercise. Here, we did not assume that a signal $x(t)$ had a Fourier series representation; we simply asked how to best approximate a given signal by a linear combination of complex exponentials, and found the resulting coefficients. These coefficients match exactly the coefficients that we obtained by assuming that the signal had a Fourier series representation, and lends some justification for the validity of the earlier analysis.

As N gets larger, the approximation error will get smaller and smaller. The question is then: will $\int_{t_0}^{t_0+T} |e(t)|^2 dt$ go to zero as N goes to infinity? If so, then the signal would, in fact, have a Fourier series representation (in the sense of having asymptotically zero error between the true signal and the approximation). It turns out that most periodic signals of practical interest will satisfy this property.

When Will a Periodic Signal Have a Fourier Series Representation?

There are various different sufficient conditions that guarantee that a given signal $x(t)$ will have a Fourier series representation. One commonly used set of conditions are known as the **Dirichlet conditions** stated as follows.

A periodic signal $x(t)$ has a Fourier series representation if all three of the following conditions are satisfied:

- The signal is *absolutely integrable* over one period:

$$\int_{t_0}^{t_0+T} |x(t)|dt < \infty.$$

- In any finite interval of time $x(t)$ has *bounded variation*, meaning that it has only a finite number of minima and maxima during any single period of the signal.
- In any finite interval of time, there are only a finite number of discontinuities, and each of these discontinuities are finite.

We won't go into the proof of why these conditions are sufficient here, but it suffices to note that signals that violate the above conditions (the last two in particular) are somewhat pathological. The first condition guarantees that the Fourier series coefficients are finite, since

$$|a_k| = \left|\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt\right| \leq \frac{1}{T}\int_{t_0}^{t_0+T} \left|x(t)e^{-jk\omega_0 t}\right| dt = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|\, dt.$$

Thus if the signal is absolutely integrable over a period, then $|a_k| < \infty$ for all $k \in \mathbb{Z}$.

Gibbs Phenomenon

If $x(t)$ has a Fourier series representation, then that representation will exactly equal $x(t)$ at all points t where $x(t)$ is continuous. At points of discontinuity in $x(t)$, the value of the Fourier series representation will be equal to the average of the values of $x(t)$ on either side of the discontinuity. One particularly interesting phenomenon occurs at points of discontinuity: the Fourier series typically overshoots the signal $x(t)$. The height of the overshoot stays constant as the number of terms N in the approximation increases, but the width shrinks. Thus, asymptotically, the error goes to zero (although technically the two signals are not exactly the same at the discontinuity).

4.4 Properties of Continuous-Time Fourier Series

We will now derive some useful properties of the Fourier series coefficients. Throughout we will use the notation

$$x(t) \stackrel{\text{FS}}{\longleftrightarrow} a_k$$

to denote that the T-periodic signal $x(t)$ has Fourier series coefficients a_k, $k \in \mathbb{Z}$.

4.4.1 Linearity

Suppose we have two signals $x_1(t)$ and $x_2(t)$, each of which is periodic with period T. Let

$$x_1(t) \overset{\text{FS}}{\longleftrightarrow} a_k, \quad x_2(t) \overset{\text{FS}}{\longleftrightarrow} b_k.$$

For any complex scalars α, β, let $g(t) = \alpha x_1(t) + \beta x_2(t)$. Then

$$g(t) \overset{\text{FS}}{\longleftrightarrow} \alpha a_k + \beta b_k.$$

The above property follows immediately from the definition of the Fourier series coefficients (since integration is linear).

Example 4.5. Consider $x_1(t) = \cos(\omega_0 t)$ and $x_2(t) = \sin(\omega_0 t)$. We have

$$\begin{aligned} g(t) = \alpha\cos(\omega_0 t) + \beta\sin(\omega_0 t) &= \frac{\alpha}{2}e^{j\omega_0 t} + \frac{\alpha}{2}e^{-j\omega_0 t} + \frac{\beta}{2j}e^{j\omega_0 t} - \frac{\beta}{2j}e^{-j\omega_0 t} \\ &= \left(\frac{\alpha}{2} + \frac{\beta}{2j}\right)e^{j\omega_0 t} + \left(\frac{\alpha}{2} - \frac{\beta}{2j}\right)e^{-j\omega_0 t}. \end{aligned}$$

Thus, we see that each Fourier series coefficient of $g(t)$ is indeed given by a linear combination of the corresponding Fourier series coefficients of $x_1(t)$ and $x_2(t)$. □

4.4.2 Time Shifting

Define $g(t) = x(t - \tau)$, where $\tau \in \mathbb{R}$ is some delay. Let the Fourier series coefficients of $g(t)$ be given by b_k, $k \in \mathbb{Z}$. Then

$$b_k = \frac{1}{T}\int_{t_0}^{t_0+T} g(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{t_0}^{t_0+T} x(t-\tau)e^{-jk\omega_0 t}dt.$$

Define $\bar{t} = t - \tau$, so that $\frac{d\bar{t}}{dt} = 1$. Then we have

$$b_k = \frac{1}{T}\int_{t_0-\tau}^{t_0-\tau+T} x(\bar{t})e^{-jk\omega_0(\bar{t}+\tau)}d\bar{t} = e^{-jk\omega_0\tau}\frac{1}{T}\int_{t_0-\tau}^{t_0-\tau+T} x(\bar{t})e^{-jk\omega_0\bar{t}}d\bar{t} = e^{-jk\omega_0\tau}a_k.$$

Thus

$$x(t-\tau) \overset{\text{FS}}{\longleftrightarrow} e^{-jk\omega_0\tau}a_k.$$

Example 4.6. Consider $x(t) = \cos(\omega_0 t)$, and let $g(t) = x(t - \tau)$ We have

$$g(t) = \cos(\omega_0(t-\tau)) = \frac{1}{2}e^{j\omega_0(t-\tau)} + \frac{1}{2}e^{-j\omega_0(t-\tau)} = \frac{1}{2}e^{-j\omega_0\tau}e^{j\omega_0 t} + \frac{1}{2}e^{j\omega_0\tau}e^{-j\omega_0 t}.$$

Thus, we see that the coefficient of $e^{j\omega_0 t}$ is $\frac{1}{2}e^{-j\omega_0\tau}$, and the coefficient of $e^{-j\omega_0 t}$ is $\frac{1}{2}e^{j\omega_0\tau}$, as predicted by the expressions derived above. □

4.4.3 Time Reversal

Define $y(t) = g(-t)$. Let the Fourier series coefficients of $g(t)$ be given by b_k, $k \in \mathbb{Z}$. Note that

$$g(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}.$$

Thus we have

$$x(-t) \overset{\text{FS}}{\longleftrightarrow} a_{-k},$$

i.e., the Fourier series coefficients for a time-reversed signal are just the time-reversal of the Fourier series coefficients for the original signal. Note that this is **not** true for the output of LTI systems (a time reversal of the input to an LTI system does not necessarily mean that the output is a time-reversal of the original output).

Example 4.7. Consider $x(t) = \sin(\omega_0 t)$ and define $g(t) = x(-t)$. First, note that

$$x(t) = \sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} = a_1 e^{j\omega_0 t} + a_{-1}e^{-j\omega_0 t}.$$

We have

$$g(t) = \sin(-\omega_0 t) = \frac{1}{2j}e^{-j\omega_0 t} - \frac{1}{2j}e^{j\omega_0 t} = b_1 e^{j\omega_0 t} + b_{-1}e^{-j\omega_0 t},$$

and thus we see that $b_1 = -\frac{1}{2j} = a_{-1}$ and $b_{-1} = \frac{1}{2j} = a_1$. □

4.4.4 Time Scaling

Consider the signal $g(t) = x(\alpha t)$, where $\alpha \in \mathbb{R}_{>0}$. Thus, $g(t)$ is a time-scaled version of $x(t)$. Note that the period of $g(t)$ is $\frac{T}{2}$, where T is the period of $x(t)$. We have

$$g(t) = x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t}.$$

Thus, $g(t)$ has the same Fourier series coefficients as $x(t)$, but the Fourier series **representation** has changed: the frequency is now $\omega_0\alpha$ rather than ω_0, to reflect the fact that the harmonic family is in terms of the new period $\frac{T}{\alpha}$ rather than T.

Example 4.8. Consider $x(t) = \cos(\omega_0 t)$, which has series representation

$$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}.$$

Then we have

$$g(t) = x(\alpha t) = \cos(\omega_0 \alpha t) = \frac{1}{2}e^{j\omega_0 \alpha t} + \frac{1}{2}e^{-j\omega_0 \alpha t}.$$

□

4.4.5 Multiplication

Let $x_1(t)$ and $x_2(t)$ be T-periodic signals with Fourier series a_k and b_k respectively. Consider the signal $g(t) = x_1(t)x_2(t)$, and note that $g(t)$ is also T-periodic. We have

$$g(t) = x_1(t)x_2(t) = \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t} = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_l b_n e^{j(l+n)\omega_0 t}.$$

Define $k = l + n$, so that

$$g(t) = \sum_{l=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_l b_{k-l} e^{jl\omega_0 t} = \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_l b_{k-l} \right) e^{jk\omega_0 t}.$$

Thus the Fourier series coefficients of $g(t)$ are given by

$$g(t) = x_1(t)x_2(t) \stackrel{\text{FS}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

In other words, the Fourier series coefficients of a product of two signals are given by the **convolution** of the corresponding Fourier series coefficients.

Example 4.9. Consider the signals $x_1(t) = \cos(\omega_0 t)$ and $x_2(t) = \sin(\omega_0 t)$. Define $g(t) = x_1(t)x_2(t)$. The Fourier series representations of $x_1(t)$ and $x_2(t)$ are given by

$$x_1(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}, \quad x_2(t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}.$$

Denote the Fourier series coefficients of $x_1(t)$ by the sequence a_k, with $a_{-1} = a_1 = \frac{1}{2}$, and $a_k = 0$ otherwise. Similarly, denote the Fourier series coefficients of $x_2(t)$ by the sequence b_k, with $b_{-1} = -\frac{1}{2j}$, $b(1) = \frac{1}{2j}$, and $b_k = 0$ otherwise. Denote the Fourier series coefficients of $g(t)$ by c_k, $k \in \mathbb{Z}$. Then we have

$$c_k = a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Convolving the two sequences given above, we see that $c_{-2} = -\frac{1}{4j}$, $c_2 = \frac{1}{4j}$, and $c_k = 0$ otherwise. Thus

$$g(t) = x_1(t)x_2(t) = \cos(\omega_0 t)\sin(\omega_0 t) = \frac{1}{4j}e^{j2\omega_0 t} - \frac{1}{4j}e^{-j2\omega_0 t}.$$

Noting that $\cos(\omega_0 t)\sin(\omega_0 t) = \frac{1}{2}\sin(2\omega_0 t)$, we see that the above expression is, in fact, correct. □

4.4.6 Parseval's Theorem

Consider a T-periodic signal $x(t)$, and let a_k, $k \in \mathbb{Z}$ be its Fourier series coefficients. Now let us consider the average power of $x(t)$ over one period, defined as

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt.$$

Substituting $|x(t)|^2 = x^*(t)x(t)$ and the Fourier series for $x(t)$ we have

$$\begin{aligned}\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt &= \frac{1}{T}\int_{t_0}^{t_0+T} \left(\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} a_n^* e^{jn\omega_0 t}\right) dt \\ &= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_k^* a_n \frac{1}{T}\int_{t_0}^{t_0+T} e^{-jk\omega_0 t} e^{jn\omega_0 t} dt.\end{aligned}$$

Using the fact that $e^{jk\omega_0 t}$ and $e^{jn\omega_0 t}$ are orthogonal for $k \neq n$, we have

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} a_k^* a_k = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

This leads to **Parseval's Theorem**: for a T-periodic signal $x(t)$ with Fourier series coefficients a_k, $k \in \mathbb{Z}$, we have

$$\boxed{\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.}$$

Example 4.10. Consider the signal $x(t) = \cos(\omega_0 t)$, with Fourier coefficients $a_1 = a_{-1} = \frac{1}{2}$ and $a_k = 0$ otherwise. We have

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt = \frac{1}{T}\int_{t_0}^{t_0+T} \cos^2(\omega_0 t) dt = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{1}{2}(1+\cos(2\omega_0 t)) dt = \frac{1}{2}.$$

We also have

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = a_1^2 + a_{-1}^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

which agrees with the direct calculation of average power, as indicated by Parseval's Theorem. □

4.5 Fourier Series for Discrete-Time Periodic Signals

We now turn our attention to the discrete-time Fourier series. Specifically, consider the discrete-time signal $x[n]$, $n \in \mathbb{Z}$, and suppose it is N-periodic. Let

$\omega_0 = \frac{2\pi}{N}$, and note that the signal $\phi_k[n] = e^{jk\omega_0 n}$ is also N-periodic for any $k \in \mathbb{Z}$. Thus, as in the case for continuous-time signals, we would like to write $x[n]$ as a linear combination of signals from the harmonic family $\phi_k[n]$, $k \in \mathbb{Z}$, i.e.,

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 n} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{N}n}.$$

At this point, we encounter the first main difference between the discrete-time and continuous-time Fourier series. Recall that a discrete-time complex exponential with frequency ω_0 is the same as a discrete-time complex exponential with frequency $\omega_0 + 2\pi$. Specifically, for any $k \in \mathbb{Z}$, consider

$$\phi_{k+N}[n] = e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} e^{jN\omega_0 n} = e^{jk\omega_0 n},$$

since $N\omega_0 = 2\pi$. Thus, there are only N different complex exponentials in the discrete-time harmonic family for the fundamental frequency ω_0, and so we have the following.

> The discrete-time Fourier series for an N-periodic signal $x[n]$ is given by
>
> $$x[n] = \sum_{k=n_0}^{n_0+N-1} a_k e^{jk\omega_0 n} = \sum_{k=n_0}^{n_0+N-1} a_k e^{jk\frac{2\pi}{N}n} \tag{4.2}$$
>
> where n_0 is any integer.

In other words, the discrete-time signal can be written in terms of any N contiguous multiples of the fundamental frequency ω_0. Since there are only N coefficients a_k in this representation, and since it does not matter which contiguous N members of the harmonic family we choose, **the Fourier series coefficients are N-periodic** as well, i.e., $a_k = a_{k+N}$ for all $k \in \mathbb{Z}$.

4.5.1 Finding the Discrete-Time Fourier Series Coefficients

To find the Fourier series coefficients in (4.2), we use a similar trick as in the continuous-time case. Specifically, first multiply both sides of (4.2) by $e^{-jr\omega_0 n}$, where r is any integer, and sum both sides over N terms. This gives

$$\sum_{n=n_1}^{n_1+N-1} x[n]e^{-jr\omega_0 n} = \sum_{n=n_1}^{n_1+N-1} \sum_{k=n_0}^{n_0+N-1} a_k e^{j(k-r)\omega_0 n},$$

where n_1 is any integer. Interchanging the summations, we have

$$\sum_{n=n_1}^{n_1+N-1} x[n]e^{-jr\omega_0 n} = \sum_{k=n_0}^{n_0+N-1} a_k \sum_{n=n_1}^{n_1+N-1} e^{j(k-r)\omega_0 n}.$$

Now suppose that $k - r$ is a multiple of N. In this case we obtain

$$\sum_{n=n_1}^{n_1+N-1} e^{j(k-r)\omega_0 n} = \sum_{n=n_1}^{n_1+N-1} 1 = N.$$

On the other hand, if $r - k$ is not a multiple of N, we use the finite sum formula to obtain

$$\sum_{n=n_1}^{n_1+N-1} e^{j(k-r)\omega_0 n} = \frac{e^{j(k-r)\omega_0 n_1} - e^{j(k-r)\omega_0 (n_1+N)}}{e^{j(k-r)\omega_0} - 1} = 0.$$

Thus, we have the following.

The discrete-time Fourier series coefficients are given by

$$a_k = \frac{1}{N} \sum_{n=n_1}^{n_1+N-1} x[n] e^{-jk\omega_0 n}$$

where n_1 is any integer. The Fourier series coefficients are N-periodic, i.e., $a_k = a_{k+N}$.

Example 4.11. Consider the N-periodic signal $x[n]$, which is equal to 1 for $-N_1 \leq n \leq N_1$ and zero otherwise (modulo the periodic constraints).

We have

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n}.$$

If $k = 0$, we have $a_0 = \frac{2N_1+1}{N}$. For $k \in \{1, 2, \ldots, N-1\}$, we have (via the finite sum formula)

$$\begin{aligned} a_k = \frac{1}{N} \frac{e^{jk\omega_0 N_1} - e^{jk\omega_0 (N_1+1)}}{1 - e^{-jk\omega_0}} &= \frac{1}{N} \frac{e^{-jk\omega_0 \frac{1}{2}}}{e^{-jk\omega_0 \frac{1}{2}}} \frac{e^{jk\omega_0 (N_1 + \frac{1}{2})} - e^{jk\omega_0 (N_1+\frac{1}{2})}}{e^{jk\omega_0 \frac{1}{2}} - e^{-jk\omega_0 \frac{1}{2}}} \\ &= \frac{1}{N} \frac{\sin(k\omega_0 (N_1 + \frac{1}{2}))}{\sin(k \frac{\omega_0}{2})}. \end{aligned}$$

□

It is of interest to note that we do not have to worry about convergence conditions for discrete-time Fourier series, as we did in the continuous-time case. Specifically, for an N-periodic discrete-time signal $x[n]$, we only require N numbers to completely specify the entire signal. The Fourier series coefficients a_k, $k \in \{0, 1, \ldots, N-1\}$ thus contain as much information as the signal itself, and form a perfect representation of the signal. In other words, for discrete-time signals, the Fourier series representation is just a transformation of the signal into another form; we do not encounter discrepancies like the Gibbs phenomenon in discrete-time.

4.5.2 Properties of the Discrete-Time Fourier Series

The discrete-time Fourier series has properties that can be derived in almost the same way as the properties for the continuous-time Fourier series (linearity, time-shifting, etc.). Here we will just discuss the multiplication property and Parseval's theorem.

Multiplication of Discrete-Time Periodic Signals

First, let's start with an example. Consider two 2-periodic signals $x_1[n]$ and $x_2[n]$. We know that the Fourier series respresentations can be uniquely specified by two coefficients. Specifically,

$$x_1[n] = a_0 + a_1 e^{j\omega_0 n}, \quad x_2[n] = b_0 + b_1 e^{j\omega_0 n}.$$

Now consider the product

$$\begin{aligned} g[n] = x_1[n]x_2[n] &= (a_0 + a_1 e^{j\omega_0 n})(b_0 + b_1 e^{j\omega_0 n}) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) e^{j\omega_0 n} + a_1 b_1 e^{j2\omega_0 n}. \end{aligned}$$

Now, note that $\omega_0 = \frac{2\pi}{2}$, and thus $e^{j2\omega_0 n} = 1$. This gives

$$g[n] = (a_0 b_0 + a_1 b_1) + (a_0 b_1 + a_1 b_0) e^{j\omega_0 n}.$$

Thus, the Fourier series coefficients of $g[n]$ are given by $c_0 = a_0 b_0 + a_1 b_1$ and $c_1 = a_0 b_1 + a_1 b_0$. We can write these in a unform way as follows:

$$\begin{aligned} c_0 &= a_0 b_0 + a_1 b_1 = a_0 b_0 + a_1 b_{-1} = \sum_{l=0}^{1} a_l b_{-l} \\ c_1 &= a_0 b_1 + a_1 b_0 = \sum_{l=0}^{1} a_l b_{1-l}, \end{aligned}$$

where we used the fact that $b_1 = b_{-1}$ by the periodic nature of the discrete-time Fourier series coefficients. The above expressions show that the Fourier series coefficients of the product of the two signals are given by a form of convolution of the coefficients of those signals; however, the convolution is over a *finite number of terms*, as opposed to over all time-indices.

Let us generalize this to functions with a larger period. Let $x_1[n]$ and $x_2[n]$ be two N-periodic discrete-time signals, with discrete-time Fourier series coefficients a_k and b_k, respectively. We have

$$x_1[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n}, \quad x_2[n] = \sum_{k=0}^{N-1} b_k e^{jk\omega_0 n}.$$

Define the signal $g[n] = x_1[n]x_2[n]$. We have

$$g[n] = x_1[n]x_2[n] = \left(\sum_{l=0}^{N-1} a_l e^{jl\omega_0 n}\right)\left(\sum_{r=0}^{N-1} b_r e^{jr\omega_0 n}\right) = \sum_{l=0}^{N-1}\sum_{r=0}^{N-1} a_l b_r e^{j(l+r)\omega_0 n},$$

where we have used l and r as the indices in the Fourier series in order to keep the terms in the two sums distinct.

Define the new variable $k = l + r$. Substituting into the above expression, this gives

$$\begin{aligned}
g[n] &= \sum_{l=0}^{N-1}\sum_{k=l}^{l+N-1} a_l b_{k-l} e^{jk\omega_0 n} \\
&= \sum_{l=0}^{N-1}\left(\sum_{k=l}^{N-1} a_l b_{k-l} e^{jk\omega_0 n} + \sum_{k=N}^{l+N-1} a_l b_{k-l} e^{jk\omega_0 n}\right) \\
&= \sum_{l=0}^{N-1}\left(\sum_{k=l}^{N-1} a_l b_{k-l} e^{jk\omega_0 n} + \sum_{k=0}^{l-1} a_l b_{k+N-l} e^{j(k+N)\omega_0 n}\right) \\
&= \sum_{l=0}^{N-1}\left(\sum_{k=l}^{N-1} a_l b_{k-l} e^{jk\omega_0 n} + \sum_{k=0}^{l-1} a_l b_{k-l} e^{jk\omega_0 n}\right) \\
&= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} a_l b_{k-l} e^{jk\omega_0 n} \\
&= \sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1} a_l b_{k-l}\right) e^{jk\omega_0 n}
\end{aligned}$$

Thus, the Fourier series coefficients of $g[n]$ are given by

$$c_k = \sum_{l=0}^{N-1} a_l b_{k-l}$$

for $0 \le k \le N-1$. The above convolution is known as the **periodic convolution** of two periodic signals; for any k, there are only N terms in the summation. We saw an example of this with the 2-periodic signals that we had above. In essence, the above calucations are simply multiplying together the discrete-time Fourier series representations of the two signals (each of which has N terms), and then using the periodicity of the discrete-time complex exponentials in the frequencies to combine terms together. Note that we can actually perform the sum over *any* continguous N values of l, since all of the signals involved are periodic.

Parseval's Theorem for Discrete-Time Signals

Let $x[n]$ be an N-periodic discrete-time signal, with Fourier series coefficients a_k, $0 \leq k \leq N-1$. The average power of $x[n]$ over one period is

$$\frac{1}{N}\sum_{n=n_0}^{n_0+N-1}|x[n]|^2,$$

where n_0 is any integer. Parseval's theorem for discrete-time signals states the following.

$$\frac{1}{N}\sum_{n=n_0}^{n_0+N-1}|x[n]|^2 = \sum_{k=0}^{N-1}|a_k|^2.$$

The following example illustrates the application of the various facts that we have seen about the discrete-time Fourier series.

Example 4.12. Suppose we are told the following facts about a discrete-time signal $x[n]$.

- $x[n]$ is periodic with period $N = 6$.
- $\sum_{n=0}^{5} x[n] = 2$.
- $\sum_{n=2}^{7}(-1)^n x[n] = 1$.
- $x[n]$ has the minimum power per period of all signals satisfying the preceding three conditions.

The above facts are sufficient for us to uniquely determine $x[n]$. First, note that the Fourier series representation of $x[n]$ is

$$x[n] = \sum_{k=0}^{5} a_k e^{jk\omega_0 n} = \sum_{k=0}^{5} a_k e^{jk\frac{\pi}{3}n},$$

where we used the fact that the signal is 6-periodic.

From the second fact, we have

$$a_0 = \frac{1}{N}\sum_{k=0}^{N-1} x[n] = \frac{2}{6} = \frac{1}{3}.$$

To use the third fact, note that $(-1)^n = e^{-j\pi n} = e^{-j3\frac{\pi}{3}n}$. Thus, the third fact seems to be related to the Fourier series coefficient for $k = 3$. Specifically, we have

$$a_3 = \frac{1}{N}\sum_{k=2}^{7} x[n]e^{-j\pi n} = \frac{1}{6}.$$

To use the last fact, note from Parseval's theorem that the power of the signal over one period is given by

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{5}|x[n]|^2 &= \sum_{k=0}^{5}|a_k|^2 \\ &= |a_0|^2+|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2+|a_5|^2.\end{aligned}$$

We are told that $x[n]$ has the minimum average power over all signals that satisfy the other three conditions. Since the other three conditions have already set a_0 and a_1, the average power is given by setting all of the other Fourier series coefficients to 0. Thus, we have

$$x[n] = \sum_{k=0}^{5} a_k e^{jk\frac{\pi}{3}n} = a_0 + a_3 e^{j\pi n} = \frac{1}{3} + \frac{1}{6}(-1)^n.$$

□

Chapter 5

The Continuous-Time Fourier Transform

Reading: *Signals and Systems*, Chapter 4.1-4.6.

In the last part of the course, we decomposed periodic signals into superpositions of complex exponentials, where each complex exponential is a member of the harmonic family corresponding to the fundamental period of the signal. We now turn our attention to the case where the signal of interest is not periodic. As we will see, the main idea will be to view an aperiodic signal as a periodic signal whose period goes to ∞.

5.1 The Fourier Transform

Suppose that we are given a signal $x(t)$ that is aperiodic. As a concrete example, suppose that $x(t)$ is a solitary square pulse, with $x(t) = 1$ if $-T_1 \leq t \leq T_1$, and zero elsewhere. Clearly $x(t)$ is not periodic.

Now define a new signal $\tilde{x}(t)$ which is a periodic extension of $x(t)$ with period T. In other words, $\tilde{x}(t)$ is obtained by repeating $x(t)$, where each copy is shifted T units in time. This $\tilde{x}(t)$ has a Fourier series representation, which we found in the last chapter to be

$$a_0 = \frac{2T_1}{T}, \quad a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T},$$

for $k \in \mathbb{Z}$.

Now recall that the Fourier series coefficients are calculated as follows:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-jk\omega_0 t}dt.$$

However, we note that $x(t) = \tilde{x}(t)$ in the interval of integration, and thus

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt.$$

Furthermore, since $x(t)$ is zero for all t outside the interval of integration, we can expand the limits of the integral to obtain

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt.$$

Let us define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

This is called the **Fourier transform** of the signal $x(t)$, and the Fourier series coefficients can be viewed as samples of the Fourier transform, scaled by $\frac{1}{T}$, i.e.,

$$a_k = \frac{1}{T}X(jk\omega_0), k \in \mathbb{Z}.$$

Now consider the fact that

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}.$$

Since $\omega_0 = \frac{2\pi}{T}$, this becomes

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$

Now consider what happens as the period T gets bigger. In this case, $\tilde{x}(t)$ approaches $x(t)$, and so the above expression becomes a representation of $x(t)$. As $T \to \infty$, we have $\omega_0 \to 0$. Since each term in the summand can be viewed as the area of the rectangle whose height is $X(jk\omega_0)e^{jk\omega_0 t}$ and whose base goes from $k\omega_0$ to $(k+1)\omega_0$, we see that as $\omega_0 \to 0$, the sum on the right hand side approaches the area underneath the curve $X(j\omega)e^{j\omega t}$ (where t is held fixed). Thus, as $T \to \infty$ we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Thus we have the following.

Given a continuous-time signal $x(t)$, the **Fourier Transform** of the signal is given by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

The **Inverse Fourier Transform** of the signal is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

The Fourier transform $X(j\omega)$ is also called the **spectrum** of the signal, as it represents the contribution of the complex exponential of frequency ω to the signal $x(t)$.

Example 5.1. Consider the signal $x(t) = e^{-at}u(t)$, $a \in \mathbb{R}_{>0}$. The Fourier transform of this signal is

$$\begin{aligned} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_0^{\infty} \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

To visualize $X(j\omega)$, we plot its magnitude and phase on separate plots (since $X(j\omega)$ is complex-valued in general). We have

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

The plots of these quantities are show in Fig. 4.5 of the text. □

Example 5.2. Consider the signal $x(t) = \delta(t)$. We have

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1.$$

In other words, the spectrum of the impulse function has an equal contribution at all frequencies. □

Example 5.3. Consider the signal $x(t)$ which is equal to 1 for $-T_1 \le t \le T_1$ and zero elsewhere. We have

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{1}{j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right) = \frac{2\sin(\omega T_1)}{\omega}.$$

□

Example 5.4. Consider the signal whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| \le W \\ 0 & |\omega| > W \end{cases}.$$

We have

$$\begin{aligned} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\frac{1}{jt}\, e^{j\omega t}\Big|_{-W}^{W} \\ &= \frac{\sin(Wt)}{\pi t}. \end{aligned}$$

□

The previous two examples showed the following. When $x(t)$ is a square pulse, then $X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$ and when $X(j\omega)$ is a square pulse, $x(t) = \frac{\sin(Wt)}{\pi t}$. This is an example of the *duality* property of Fourier transforms, which we will see later.

Functions of the form $\frac{\sin(Wt)}{\pi t}$ will show up frequently, and are called **sinc** functions. Specifically

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}.$$

Thus $\frac{2\sin(\omega T_1)}{\omega} = 2T_1\text{sinc}\left(\frac{\omega T_1}{\pi}\right)$ and $\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$.

5.1.1 Existence of Fourier Transform

Just as we saw with the Fourier series for periodic signals, there are some rather mild conditions under which a signal $x(t)$ is guaranteed to have a Fourier transform (such that the inverse Fourier transform converges to the true signal). Specifically, there are a set of sufficient conditions (also called Dirichlet conditions) under which a continuous-time signal $x(t)$ is guaranteed to have a Fourier transform:

1. $x(t)$ is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)|dt < \infty$.

2. $x(t)$ has a finite number of maxima and minima in any finite interval.

3. $x(t)$ has a finite number of discontinuities in any finite interval, and each of these discontinuities is finite.

If all of the above conditions are satisfied, $x(t)$ is guaranteed to have a Fourier transform. Note that this only a *sufficient* set of conditions, and not necessary.

An alternate sufficient condition is that the signal have finite energy (i.e., that it be **square integrable**):

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty.$$

For example, the signal $x(t) = \frac{1}{t}u(t-1)$ is square integrable, but not absolutely integrable. Thus the finite energy condition guarantees that $x(t)$ will have a Fourier transform, whereas the Dirichlet conditions do not apply.

5.2 Fourier Transform of Periodic Signals

The Fourier transform can also be applied to certain periodic signals (although such signals will not be absolutely integrable or square integrable over the entire time-axis). A direct application of the Fourier transform equation to such signals will not necessarily yield a meaningful answer, due to the fact that periodic signals do not die out. Instead, we will work backwards by starting with a frequency domain signal and doing an inverse Fourier transform to see what pairs arise.

Thus, consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = 2\pi\delta(\omega - \omega_0),$$

i.e., the frequency domain signal is a single impulse at $\omega = \omega_0$, with area 2π. Using the inverse Fourier transform, we obtain

$$\begin{aligned} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega &= \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega \\ &= e^{j\omega_0 t}. \end{aligned}$$

Thus, the Fourier transform of $x(t) = e^{j\omega_0 t}$ is $X(j\omega) = 2\pi\delta(\omega - \omega_0)$. Similarly, if

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0),$$

then an application of the inverse Fourier transform gives

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

In other words, if $x(t)$ is a periodic signal with Fourier series coefficients a_k, then the Fourier transform of $x(t)$ consists of a sequence of impulse functions, each spaced at multiples of ω_0; the area of the impulse at $k\omega_0$ will be $2\pi a_k$.

Example 5.5. Consider the signal $x(t) = \cos(\omega_0 t)$. The Fourier series coefficients are $a_1 = a_{-1} = \frac{1}{2}$. Thus, the Fourier transform of this signal is given by

$$X(j\omega) = a_1 2\pi\delta(\omega - \omega_0) + a_{-1}2\pi\delta(\omega + \omega_0) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).$$

□

Example 5.6. Consider the periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

The Fourier series coefficients for this signal are given by

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT)dt = \frac{1}{T} \\ a_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} \delta(t - nT)e^{-jk\omega_0 t}dt = \frac{1}{T}. \end{aligned}$$

Thus, the Fourier transform of this signal is given by

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2k\pi}{T}\right).$$

Thus, if $x(t)$ is an impulse train with period T, its Fourier transform is also an impulse train in the frequency domain, except with period $\frac{2\pi}{T}$. Once again, we see that if T increases (i.e., the period increases in the time-domain) we obtain a time-shrinking in the frequency domain. □

5.3 Properties of the Continuous-Time Fourier Transform

We will now discuss various properties of the Fourier transform. As with the Fourier series, we will find it useful to introduce the following notation. Suppose $x(t)$ is a time-domain signal, and $X(j\omega)$ is its Fourier transform. We then say

$$\begin{aligned} X(j\omega) &= \mathcal{F}\{x(t)\} \\ x(t) &= \mathcal{F}^{-1}\{X(j\omega)\}. \end{aligned}$$

We will also use the notation

$$x(t) \overset{\mathrm{F}}{\longleftrightarrow} X(j\omega)$$

to indicate that $x(t)$ and $X(j\omega)$ are Fourier transform pairs.

5.3.1 Linearity

The first property of Fourier transforms is easy to show:

$$\mathcal{F}\{\alpha x_1(t) + \beta x_2(t)\} = \alpha\mathcal{F}\{x_1(t)\} + \beta\mathcal{F}\{x_2(t)\},$$

which follows immediately from the definition of the Fourier transform.

5.3.2 Time-Shifting

Suppose $x(t)$ is a signal with Fourier transform $X(j\omega)$. Define $g(t) = x(t-\tau)$ where $\tau \in \mathbb{R}$. Then we have

$$G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t-\tau)e^{-j\omega t}dt = e^{-j\omega\tau}X(j\omega).$$

Thus

$$\mathcal{F}\{x(t-\tau)\} = e^{-j\omega\tau}X(j\omega).$$

Note the implication: if we time-shift a signal, the magnitude of its Fourier transform is not affected. Only the phase of the Fourier transform gets shifted by $-\omega\tau$ at each frequency ω.

5.3.3 Conjugation

Consider a signal $x(t)$. We have

$$X^*(j\omega) = \left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt.$$

Thus,

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \mathcal{F}\{x^*(t)\}.$$

The above is true for any signal $x(t)$ that has a Fourier transform. Now suppose additionally that $x(t)$ is a real-valued signal. Then we have $x^*(t) = x(t)$ for all $t \in \mathbb{R}$. Thus

$$X^*(-j\omega) = \mathcal{F}\{x^*(t)\} = \mathcal{F}\{x(t)\} = X(j\omega).$$

Based on the above relationship between $X^*(-j\omega)$ and $X(j\omega)$ for real-valued signals, we see the following. Write $X(j\omega)$ in polar form as

$$X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}.$$

Then we have

$$X(-j\omega) = X^*(j\omega) = |X(j\omega)|e^{-j\angle X(j\omega)}.$$

Thus, for any ω, $X(-j\omega)$ has the same magnitude as $X(j\omega)$, and the phase of $X(-j\omega)$ is the negative of the phase of $X(j\omega)$. Thus, when plotting $X(j\omega)$, we only have to plot the magnitude and phase for positive values of ω, as the plots for negative values of ω can be easily recovered according to the relationships described above.

Example 5.7. Consider again the signal $x(t) = e^{-at}u(t)$; we saw earlier that the Fourier transform of this signal is

$$X(j\omega) = \frac{1}{a + j\omega}.$$

It is easy to verify that

$$X(-j\omega) = \frac{1}{a - j\omega} = X^*(j\omega),$$

as predicted. Furthermore, we can see from the plots of the magnitude and phase of $X(j\omega)$ that the magnitude is indeed an even function, and the phase is an odd function. □

Suppose further that $x(t)$ is even (in addition to being real-valued). Then we have $x(t) = x(-t)$. Then we have

$$\begin{aligned} X(-j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = \int_{-\infty}^{\infty} x(-t)e^{j\omega t}dt &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= X(j\omega). \end{aligned}$$

This, together with the fact that $X(-j\omega) = X^*(j\omega)$ for real-valued signals indicates that $X(j\omega)$ is real-valued and even.

Similarly, if $x(t)$ is real-valued and odd, we have $X(j\omega)$ is purely imaginary and odd.

Example 5.8. Consider the signal $x(t) = e^{-a|t|}$, where a is a positive real number. This signal is real-valued and even. We have

$$\begin{aligned} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt &= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2}. \end{aligned}$$

As predicted, $X(j\omega)$ is real-valued and even. □

5.3.4 Differentiation

Consider the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Differentiating both sides with respect to t, we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

Thus, we see that

$$\mathcal{F}\{\frac{dx(t)}{dt}\} = j\omega X(j\omega).$$

5.3.5 Time and Frequency Scaling

Let a be a nonzero real number and consider the signal $g(t) = x(at)$ (i.e., a time-scaling of $x(t)$). We have

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt.$$

If we perform the substitution $\tau = at$, we have

$$\mathcal{F}\{g(t)\} = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\tau, & a > 0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\tau, & a < 0 \end{cases}.$$

This can be written in a uniform way as

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}X\left(j\frac{\omega}{a}\right).$$

Thus we see again that shrinking a signal in the time-domain corresponds to expanding it in the frequency domain, and vice versa.

5.3.6 Duality

We have already seen a few examples of the *duality* property: suppose $x(t)$ has Fourier transform $X(j\omega)$. Then if we have a time-domain signal that has the same form as $X(j\omega)$, the Fourier transform of that signal will have the same form as $x(t)$. For example, the square pulse in the time-domain had a Fourier transform in the form of a sinc function, and a sinc function in the time-domain had a Fourier transform in the form of a square pulse.

We can consider another example. Suppose $x(t) = e^{-|t|}$. Then one can verify that

$$\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+\omega^2}.$$

Specifically we have

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\omega^2}e^{-j\omega t}d\omega$$

If we multiply both sides by 2π and interchange ω and t, we obtain

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+t^2} e^{-j\omega t} dt$$

Thus, we have

$$\mathcal{F}\{\frac{2}{1+t^2}\} = 2\pi e^{-|\omega|}.$$

Duality also applies to properties of the Fourier transform. For example, recall that differentiation in the time-domain corresponds to multiplication by $j\omega$ in the frequency domain. We will now see that differentiation the frequency domain corresponds to multiplication by a certain quantity of t in the time-domain. We have

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t} dt.$$

Thus, differentiation in the frequency domain corresponds to multiplication by $-jt$ in the time-domain.

5.3.7 Parseval's Theorem

Just as with periodic signals, we have the following.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

To derive this, note that

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega\tau} d\omega dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega
\end{aligned}$$

5.3.8 Convolution

Consider a signal $x(t)$ with Fourier transform $X(j\omega)$. From the inverse Fourier transform, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega.$$

This has the interpretation that $x(t)$ can be written as a superposition of complex exponentials (with frequencies spanning the entire real axis). From earlier in the course, we know that if the input to an LTI system is $e^{j\omega t}$, then the output will be $H(j\omega)e^{j\omega t}$, where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

In other words, $H(j\omega)$ is the Fourier transform of the impulse response $h(t)$. This, together with the LTI property of the system, implies that

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega = y(t).$$

Thus, we see that the Fourier transform of the output $y(t)$ is given by

$$Y(j\omega) = H(j\omega)X(j\omega).$$

In other words:

The Fourier transform of the output of an LTI system is given by the product of the Fourier transforms of the input and the impulse response.

This is potentially the most important fact pertaining to LTI systems and frequency domain analysis. Let's derive this another way just to reinforce the fact.

Suppose that we have two signals $x(t)$ and $h(t)$, and define

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

We have

$$\begin{aligned} Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau \\ &= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= H(j\omega)X(j\omega). \end{aligned}$$

In the third line, we used the time-shifting property of the Fourier transform. Thus we see that convolution of two signals in the time-domain corresponds to multiplication of the signals in the frequency domain, i.e.,

$$\mathcal{F}\{x(t) * h(t)\} = \mathcal{F}\{x(t)\}\mathcal{F}\{h(t)\}.$$

One thing to note here pertains to the existence of the Fourier transform of $h(t)$. Specifically, recall that the LTI system is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty.$$

This is precisely the first condition in the Dirichlet conditions; thus, as long as the system is stable and the impulse response satisfies the other two conditions (which almost all real systems would), the Fourier transform is guaranteed to exist. If the system is unstable, we will need the machinery of Laplace transforms to analyze the input-output behavior, which we will defer to a later discussion.

The convolution - multiplication property is also very useful for analysis of interconnected linear systems. For example, consider the series interconnection shown in Fig. 5.1.

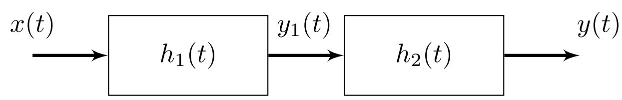


Figure 5.1: A series interconnection of systems.

We have

$$y(t) = y_1(t) * h_2(t) = (x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t)).$$

Taking Fourier transforms, we obtain

$$Y(j\omega) = X(j\omega)H_1(j\omega)H_2(j\omega).$$

This reinforces what we saw earlier, that the series interconnection of LTI systems can be lumped together in a single LTI system whose impulse response is the convolution of the impulse responses of the individual systems. In the frequency domain, their Fourier transforms get multiplied together.

One of the important implications of the convolution property is that it allows us to investigate the effect of systems on signals in the frequency domain. For example, this facilitates the design of appropriate *filters* for signals, as illustrated in the following example.

Example 5.9. Consider a signal $v(t)$ which represents a measurement of some relatively low frequency content (such as a voice signal). Suppose that we measure this signal via a microphone, whose output is given by

$$x(t) = v(t) + n(t)$$

where $n(t)$ is high-frequency noise. Note that $X(j\omega) = V(j\omega) + N(j\omega)$. We would like to take the measured signal $x(t)$ and remove the noise; unfortunately we do not have access to $n(t)$ to subtract it out. Instead, we can work in the frequency domain. Suppose we design a filter (an LTI system) whose impulse response as the following Fourier transform:

$$H(j\omega) = \begin{cases} 1 & |\omega| \leq W \\ 0 & |\omega| > W \end{cases},$$

where W is the highest frequency of the underlying signal $v(t)$. If we feed $x(t)$ into this filter, the output will have Fourier transform given by

$$Y(j\omega) = X(j\omega)H(j\omega) = V(j\omega)H(j\omega) + N(j\omega)H(j\omega).$$

If all of the frequency content of the noise $n(t)$ occurs at frequencies larger than W, then we see that $N(j\omega)H(j\omega) = 0$, and thus

$$Y(j\omega) = V(j\omega)H(j\omega) = V(j\omega).$$

In other words, we have recovered the voice signal $v(t)$ by passing $x(t)$ through the low-pass filter.

Recall that the inverse Fourier transform of the given $H(j\omega)$ is

$$h(t) = \frac{\sin(Wt)}{\pi t}.$$

However, there are various challenges with implementing an LTI system with this impulse response. One is that this is noncausal, and thus one must potentially include a sufficiently large delay (followed by a truncation of the signal) in order to apply it. Another problem is that it contains many oscillations, which may not be desirable for an impulse response.

Instead of the above filter, suppose consider another filter whose impulse response is

$$h_2(t) = e^{-at}u(t).$$

This filter can be readily implemented with an RC circuit (with the input signal being applied as an input voltage, and the output signal being the voltage across the capacitor). The Fourier transform of this impulse response is

$$H_2(j\omega) = \frac{1}{j\omega + a}.$$

The magnitude plot of this Fourier transform has content at all frequencies, and thus this filter will not completely eliminate all of the high frequency noise. However, by tuning the value of a, one can adjust how much of the noise affects the filtered signal. Note that this filter will also introduce phase shifts at different frequencies, which will also cause some distortion of the recovered signal. □

5.3.9 Multiplication

We just saw that multiplication in the time domain corresponds to convolution in the frequency domain. By duality, we obtain that multiplication in the frequency domain corresponds to convolution in the time-domain. Specifically, consider two signals $x_1(t)$ and $x_2(t)$, and define $g(t) = x_1(t)x_2(t)$. Then we have

$$\begin{aligned} G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt &= \int_{-\infty}^{\infty} x_1(t)x_2(t)e^{-j\omega t}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} x_2(t)\int_{-\infty}^{\infty} X_1(j\theta)e^{j\theta t}d\theta e^{-j\omega t}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)\int_{-\infty}^{\infty} x_2(t)e^{-j(\omega-\theta)t}dtd\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)X_2(j(\omega-\theta))d\theta. \end{aligned}$$

Thus,

$$\mathcal{F}\{x_1(t)x_2(t)\} = \frac{1}{2\pi}\left(X_1(j\omega) * X_2(j\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)X_2(j(\omega-\theta))d\theta.$$

Multiplication of one signal $x_1(t)$ by another signal $x_2(t)$ can be viewed as *modulating* the amplitude of one signal by the other. This plays a key role in communication systems.

Example 5.10. Consider a signal $s(t)$ whose frequency spectrum lies in some interval $[-W, W]$. Consider the signal $p(t) = \cos(\omega_0 t)$. The Fourier transform of $p(t)$ is given by

$$P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0).$$

Now consider the signal $x(t) = s(t)p(t)$, with Fourier transform given by

$$\begin{aligned} X(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta \\ &= \frac{1}{2}\int_{-\infty}^{\infty} S(j\theta)\delta(\omega-\theta-\omega_0)d\theta \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} S(j\theta)\delta(\omega-\theta+\omega_0)d\theta \\ &= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0)). \end{aligned}$$

Thus, multiplying the signal $s(t)$ by $p(t)$ results in a signal $x(t)$ whose frequency spectrum consists of two copies of the spectrum of $s(t)$, cenetered at the frequencies ω_0 and $-\omega_0$ and scaled by $\frac{1}{2}$. □

The above example illustrates the principle behind amplitude modulation (AM) in communication and radio systems. A low frequency signal (such as voice) is amplitude modulated to a higher frequency that is reserved for that signal. It is then transmitted at that frequency to the receiver. The following example illustrates how the receiver can recover the transmitted signal.

Example 5.11. Consider the signal $x(t) = s(t)p(t)$ from the previous example. Its frequency spectrum has two copies of the spectrum of $s(t)$, located at $\pm\omega_0$. We want to recover the original signal $s(t)$ from $x(t)$. To do this, suppose we multiply $x(t)$ by $\cos(\omega_0 t)$ again, to obtain

$$y(t) = x(t)\cos(\omega_0 t).$$

As above, we have

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0)).$$

By drawing this, we see that the frequency spectrum of $y(t)$ contains three copies of the spectrum of $s(t)$: one copy centered at $\omega = 0$ (and scaled by $\frac{1}{2}$), one copy at $2\omega_0$ scaled by $\frac{1}{4}$, and one copy at $-2\omega_0$, scaled by $\frac{1}{4}$. Thus, if we want to recover $s(t)$ from $y(t)$, we simply apply a low pass filter to it (and scale it by 2). $\square$

Chapter 6

The Discrete-Time Fourier Transform

Reading: *Signals and Systems*, Chapter 5.1-5.5.

We now turn our attention to discrete-time aperiodic signals. We saw that the Fourier transform for continuous-time aperiodic signals can be obtained by taking the Fourier series of an appropriately defined periodic signal (and letting the period go to ∞); we will follow an identical argument for discrete-time aperiodic signals. The differences between the continuous-time and discrete-time Fourier series (e.g., that the latter only involves a finite number of complex exponentials) will be reflected as differences between the continuous-time and discrete-time Fourier transforms as well.

6.1 The Discrete-Time Fourier Transform

Consider a general signal $x[n]$ which is nonzero on some interval $-N_1 \leq n \leq N_2$ and zero elsewhere. We create a periodic extension $\tilde{x}[n]$ of this signal with period N (where N is large enough so that there is no overlap). As $N \to \infty$, $\tilde{x}[n]$ becomes equal to $x[n]$ for each finite value of n.

Since $\tilde{x}[n]$ is periodic, it has a discrete-time Fourier series representation given by

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n},$$

where $\omega_0 = \frac{2\pi}{N}$. The Fourier series coefficients are given by

$$a_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} \tilde{x}[n] e^{-j\omega_0 kn},$$

where n_0 is any integer. Suppose we choose n_0 so that the interval $[-N_1, N_2]$ is contained in $[n_0, n_0+N-1]$. Then since $\tilde{x}[n] = x[n]$ in this interval, we have

$$a_k = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j\omega_0 kn} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_0 kn}.$$

Let us now define the **discrete-time Fourier transform** as

$$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

From this, we see that $a_k = \frac{1}{N}X(e^{jk\omega_0})$, i.e., the discrete-time Fourier series coefficients are obtained by sampling the discrete-time Fourier transform at periodic intervals of ω_0. Also note that $X(e^{j\omega})$ is periodic in ω with period 2π (since $e^{-j\omega n}$ is 2π-periodic).

Using the Fourier series representation of $\tilde{x}[n]$, we now have

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{jk\omega_0})e^{jk\omega_0 n} = \frac{1}{2\pi}\sum_{k=0}^{N-1} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0.$$

Once again, we see that each term in the summand represents the area of a rectangle of width ω_0 obtained from the curve $X(e^{j\omega})e^{j\omega}$. As $N \to \infty$, we have $\omega_0 \to 0$. In this case, the sum of the areas of the rectangles approaches the integral of the curve $X(e^{j\omega})e^{j\omega n}$, and since the sum was over only N samples of the function, the integral is only over one interval of length 2π. Since $\tilde{x}[n]$ approaches $x[n]$ as $N \to \infty$, we have

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

This is the **inverse discrete-time Fourier transform**, or the **synthesis equation**.

The main differences between the discrete-time and continuous-time Fourier transforms are the following. (1) The discrete-time Fourier transform $X(e^{j\omega})$ is periodic in ω with period 2π, whereas the continuous-time Fourier transform is not necessarily periodic. (2) The synthesis equation for the discrete-time Fourier transform only involves an integral over an interval of length 2π, whereas the one for the continuous-time Fourier transform is over the entire ω axis. Both of these are due to the fact that $e^{j\omega n}$ is 2π-periodic in ω, whereas the continuous-time complex exponential is not.

Since the frequency spectrum of $X(e^{j\omega})$ is only uniquely specified over an interval of length 2π, we have to be careful about what we mean by "high" and "low" frequencies. Recalling the discussion of discrete-time complex exponentials, high-frequency signals in discrete-time have frequencies close to odd multiples of π, whereas low-frequency signals have frequencies close to even multiples of π.

Example 6.1. Consider the signal

$$x[n] = a^n u[n], \quad |a| < 1.$$

We have

$$\begin{aligned} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\ &= \frac{1}{1 - ae^{-j\omega}}. \end{aligned}$$

If we plot the magnitude of $X(e^{j\omega})$, we see an illustration of the "high" versus "low" frequency effect. Specifically, if $a > 0$ then the signal $x[n]$ does not have any oscillations and $|X(e^{j\omega})|$ has its highest magnitude around even multiples of π. However, if $a < 0$, then the signal $x[n]$ oscillates between positive and negative values at each time-step; this "high-frequency" behavior is captured by the fact that $|X(e^{j\omega})|$ has its largest magnitude near odd multiples of π. See Figure. 5.4 in OW for an illustration. □

Example 6.2. Consider the signal

$$x[n] = a^{|n|}, \quad |a| < 1.$$

We have

$$\begin{aligned} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} \\ &= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \sum_{n=1}^{\infty} a^n e^{j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \frac{ae^{j\omega}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}} \\ &= \frac{1 - a^2}{1 - 2a\cos(\omega) + a^2}. \end{aligned}$$

□

6.2 The Fourier Transform of Discrete-Time Periodic Signals

In the last chapter, we saw that if we take the Fourier transform of a continuous-time periodic signal, we obtain scaled impulses located at the harmonic frequencies. We will see something similar here for discrete-time periodic signals.

First, consider the signal

$$x[n] = e^{j\omega_0 n}.$$

We claim that the Fourier transform of this signal is

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l),$$

i.e., a set of impulse functions spaced 2π apart on the frequency axis. To verify this, note that the inverse Fourier transform is given by

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

The integral is only over an interval of length 2π, and there is at most one impulse function from $X(e^{j\omega})$ in any such interval. Let that impulse be located at $\omega_0 + 2\pi r$ for some $r \in \mathbb{Z}$. Then we have

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\omega-\omega_0-2\pi r)e^{j\omega n}d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}.$$

Thus consider a periodic discrete-time signal $x[n]$, with Fourier series

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} \quad = a_0 + a_1 e^{j\omega_0 n} + a_2 e^{j2\omega_0 n} + \cdots + a_{N-1}e^{j(N-1)\omega_0 n},$$

where $\omega_0 = \frac{2\pi}{N}$. The Fourier transform of each term of the form $a_k e^{jk\omega_0 n}$ is a set of impulses spaced 2π apart, with one located at $\omega = k\omega_0$. Furthermore each of these impulses is scaled by $a_k 2\pi$. Since $a_k = a_{k+Nl}$ for any l (by the periodicity of the discrete-time Fourier series coefficients), when we add up the Fourier transforms of all of the terms in the Fourier series expansion of $x[n]$, we obtain

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

Thus, the Fourier transform of a discrete-time periodic signal is indeed a sequence of impulses located at multiples of ω_0, with a period of 2π.

Example 6.3. Consider the impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

The Fourier series coefficients of $x[n]$ are given by

$$a_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jk\omega_0 n} = \frac{1}{N}.$$

Thus the Fourier transform of $x[n]$ is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta \left(\omega - \frac{2\pi k}{N} \right) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta \left(\omega - \frac{2\pi k}{N} \right).$$

□

6.3 Properties of the Discrete-Time Fourier Transform

6.3.1 Periodicity

The Fourier transform of a signal $x[n]$ is periodic in frequency, with period 2π:

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)}).$$

This comes out of the fact that discrete-time complex exponentials are periodic in frequency with period 2π.

6.3.2 Linearity

It is easy to see that

$$\mathcal{F}\{\alpha x_1[n] + \beta x_2[n]\} = \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega}).$$

6.3.3 Time and Frequency Shifting

We have

$$\mathcal{F}\{x[n - n_0]\} = e^{-j\omega n_0} X(e^{j\omega}),$$

and

$$\mathcal{F}\{e^{j\omega_0 n} x[n]\} = X(e^{j(\omega-\omega_0)}).$$

The first property is easily proved using the inverse Fourier transform equation, and the second property is proved using the Fourier transform equation.

Example 6.4. Consider a discrete-time low-pass filter, whose Fourier transform $H_{lp}(e^{j\omega})$ is a square pulse centered at even multiples of π. Now consider the high pass filter $H_{hp}(e^{j\omega})$ which consists of square pulses centered at odd multiples of π. We see that $H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$. Thus we have

$$h_{hp}[n] = e^{j\pi n} h_{lp}[n] = (-1)^n h_{lp}[n].$$

□

6.3.4 First Order Differences

Consider the discrete-time analog of differentiation, which is to take the differences between subsequent samples of the signal. By applying the linearity and time-shifting properties, we have

$$\mathcal{F}\{x[n] - x[n-1]\} = X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega}) = (1 - e^{-j\omega})X(e^{j\omega}).$$

6.3.5 Conjugation

For any discrete-time signal (that has a Fourier transform), we have

$$\mathcal{F}\{x^*[n]\} = X^*(e^{-j\omega}).$$

Furthermore, if $x[n]$ is real, we have $x^*[n] = x[n]$ and thus

$$X(e^{j\omega}) = X^*(e^{-j\omega}).$$

6.3.6 Time-Reversal

Consider the time-reversed signal $x[-n]$. We have

$$x[-n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{-j\omega n}d\omega = \frac{1}{2\pi}\int_0^{2\pi} X(e^{-j\omega})e^{j\omega n}d\omega$$

which is obtained by performing a change of variable $\omega \to -\omega$ (note that the negative sign introduced by this change is canceled out by the reversal of the bounds of integration that arise because of the negation). Thus, we have

$$\mathcal{F}\{x[-n]\} = X(e^{-j\omega}).$$

Together with the conjugation property, we see that for real-valued even signals (where $x[n] = x[-n]$), we have

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \mathcal{F}\{x[-n]\} = X(e^{-j\omega}).$$

Thus, the Fourier transform of real, even signals is also real and even.

6.3.7 Time Expansion

Recall that for a continuous-time signal $x(t)$ and a scalar $a \neq 0$, we had

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

Thus, an expansion in the time-domain led to a compression in the frequency domain and vice versa.

In discrete-time, expansion and contraction of time-domain signals is not achieved simply by scaling the time-variable. First, since the time index must be an integer, it does not make sense to consider the signal $x[an]$, where $a < 1$. Similarly, if we consider integer values of a larger than 1, then the signal $x[an]$ only considers the values of $x[n]$ at integer multiples of a, and all information between those values is lost.

A different expansion of signals that preserves all their values is as follows. For a given signal $x[n]$ and positive integer k, define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases}.$$

Thus, the signal $x_{(k)}[n]$ is obtained by spreading the points of $x[n]$ apart by k samples and placing zeros between the samples. We have

$$\mathcal{F}\{x_{(k)}[n]\} = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk]e^{-j\omega rk}$$

since $x_{(k)}[n]$ is nonzero only at integer multiples of k. Since $x_{(k)}[rk] = x[r]$, we have

$$\mathcal{F}\{x_{(k)}[n]\} = \sum_{r=-\infty}^{\infty} x[r]e^{-j\omega rk} = X(e^{jk\omega})$$

Example 6.5. Consider the signal

$$x[n] = \begin{cases} 1 & n \in \{0, 2, 4\} \\ 2 & n \in \{1, 3, 5\} \\ 0 & \text{otherwise} \end{cases}.$$

We note that we can write $x[n]$ as

$$x[n] = g[n] + 2g[n-1],$$

where

$$g[n] = \begin{cases} 1 & n \in \{0, 2, 4\} \\ 0 & \text{otherwise} \end{cases}.$$

This $g[n]$ can be viewed as an expansion of the signal

$$h[n] = \begin{cases} 1 & 0 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Specifically, $g[n] = h_{(2)}[n]$. The Fourier transform of $h[n]$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{2} e^{-j\omega n} = \frac{1 - e^{-3j\omega}}{1 - e^{-j\omega}} = e^{-j\omega} \frac{\sin(\frac{3}{2}\omega)}{\sin(\frac{\omega}{2})}.$$

Thus, the Fourier transform of $g[n]$ is given by

$$G(e^{j\omega}) = H(e^{j2\omega}) = e^{-2j\omega} \frac{\sin(3\omega)}{\sin(\omega)}$$

Finally,

$$X(e^{j\omega}) = G(e^{j\omega}) + 2e^{-j\omega} G(e^{j\omega}) = e^{-2j\omega}(1 + 2e^{-j\omega}) \frac{\sin(3\omega)}{\sin(\omega)}.$$

□

6.3.8 Differentiation in Frequency

Suppose $x[n]$ has Fourier transform $X(e^{j\omega})$. Then we have

$$\frac{d}{d\omega} X(e^{j\omega}) = \frac{d}{d\omega} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (-jn) x[n] e^{-j\omega n}.$$

Thus

$$\mathcal{F}\{nx[n]\} = j \frac{dX(e^{j\omega})}{d\omega}.$$

6.3.9 Parseval's Theorem

We have

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega.$$

6.3.10 Convolution

Just as in continuous time, the discrete time signal $e^{j\omega n}$ is an eigenfunction of discrete-time LTI systems. Specifically, if $e^{j\omega n}$ is applied to a (stable) LTI system with impulse response $h[n]$, the output of the system will be $H(e^{j\omega})e^{j\omega n}$.

Thus consider a signal $x[n]$ written in terms of its Fourier transform as

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

This is a linear combination of complex exponentials (where the scaling factor on the complex exponential $e^{j\omega n}$ is $\frac{1}{2\pi}X(e^{j\omega})$. By the LTI property, we thus have

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \to \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})H(e^{j\omega})e^{j\omega n}d\omega = y[n].$$

The expression on the right hand side is the output $y[n]$ of the system when the input is $x[n]$. Thus we have

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

As in the continuous-time case, convolution in the time-domain is given by multiplication in the frequency domain.

Example 6.6. Consider the system shown in Fig. 5.18a in OW. Let us analyze the relationship between $y[n]$ and $x[n]$ for that system.

First, we have $w_1[n] = (-1)^n x[n] = e^{j\pi n}x[n]$. By the frequency shifting property, we see that $W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$. Next, we have

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})W_1(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)}).$$

The signal $w_3[n]$ is given by $w_3[n] = (-1)^n w_2[n]$, and thus $W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)})$. Putting this together with the expression for $W_2(e^{j\omega})$, we obtain

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}).$$

From the bottom path, we have $W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$. Thus, we have

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = \left(H_{lp}(e^{j\omega}) + H_{lp}(e^{j(\omega-\pi)})\right)X(e^{j\omega}).$$

Recall that $H_{lp}(e^{j(\omega-\pi)})$ is a high-pass filter centered at π. Thus, this system acts as a *bandstop* filter, blocking all frequencies in a certain range and letting all of the low and high frequency signals through. □

6.3.11 Multiplication

Consider two signals $x_1[n]$ and $x_2[n]$, and define $g[n] = x_1[n]x_2[n]$. The discrete-time Fourier transform of $g[n]$ is given by

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n}.$$

Replacing $x_1[n]$ by the synthesis equation

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta,$$

where we simply replaced the dummy variable ω with θ to avoid confusion, we obtain

$$\begin{aligned}
G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta x_2[n]e^{-j\omega n} \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\theta)n}d\theta \\
&= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.
\end{aligned}$$

This resembles the typical convolution of the signals $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, except that the integral is over only an interval of length 2π as opposed over the entire frequency axis. This is called the *periodic convolution* of the two signals. Recall that we saw the same thing when we considered the discrete-time Fourier series of the product of two periodic discrete-time signals.

Example 6.7. Consider the two signals

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}, \quad x_2[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n}.$$

The Fourier transforms of these signals are square pulses, where the pulse centered at 0 extend from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (for $X_1(e^{j\omega})$) and from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$ (for $X_2(e^{j\omega})$). The Fourier transform of $g[n] = x_1[n]x_2[n]$ is given by

$$G(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.$$

Since we can choose any interval of length 2π to integrate over, let's choose the interval $[-\pi, \pi)$ for convenience. We also only need to determine the values of the Fourier transform for values of ω between $-\pi$ and π, since the transform is periodic. Depending on the value of ω, there are different cases that occur:

- If $-\pi \leq \omega < -\frac{3\pi}{4}$, then there is no overlap in the signals $X_1(e^{j\theta})$ and $X_2(e^{j(\omega-\theta)})$, and thus $G(j\omega)$ is zero.
- If $-\frac{3\pi}{4} \leq \omega < -\frac{\pi}{4}$, then there is partial overlap in the signals; the product is a rectangle with support from $-\frac{\pi}{2}$ to $\omega + \frac{\pi}{4}$, and thus $G(j\omega)$ evaluates to $\frac{1}{2\pi}(\omega + \frac{3\pi}{4})$.
- If $-\frac{\pi}{4} \leq \omega < \frac{\pi}{4}$, there is full overlap and $G(j\omega)$ is $\frac{1}{2\pi}(\frac{\pi}{2}) = \frac{1}{4}$.
- If $\frac{\pi}{4} \leq \omega < \frac{3\pi}{4}$, there is partial overlap and $G(j\omega)$ is $\frac{1}{2\pi}(\frac{3\pi}{4} - \omega)$.

- If $\frac{3\pi}{4} \leq \omega < \pi$, there is no overlap and $G(j\omega)$ is zero.

Note that since we are only integrating over θ between $-\pi$ and π, the values of $X(e^{j\theta})$ outside of that interval does not matter. Thus, we could also create a new signal $\hat{X}_1(e^{j\theta})$ which is equal to $X_1(e^{j\theta})$ over the interval $[-\pi, \pi)$ and zero everywhere else. The Fourier transform can then be written as

$$G(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta,$$

i.e., it is the usual convolution of the signals $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$. □

Chapter 7

Sampling

Thus far, we have considered continuous-time signals and discrete-time signals (and their associated Fourier transforms) as two parallel tracks. In this part of the course, we will bring these two threads together and relate their frequency spectra. The main tool that we will leverage is *sampling* continuous-time signals to yield discrete-time signals. In particular, it is often desirable to process signals using digital systems (e.g., computers or embedded devices). Thus, we take a continuous-time signal, sample it at a sufficiently fast rate, process it using a digital filter, and then convert the processed signal back to a continuous-time signal.

7.1 The Sampling Theorem

Consider a continuous-time signal $x(t)$. A *sampled* version of this signal is obtained by considering the values of the signal only at certain discrete points in time. In particular, *periodic* or *uniform* sampling occurs when we pick some positive real number T_s, and consider the values $x(nT_s)$, $n \in \mathbb{Z}$. We will often denote this discrete sequence as $x[n]$ (dropping the sampling period T_s), which was the notation that we used in our analysis of discrete-time signals. The *sampling frequency* is denoted by $\omega_s = \frac{2\pi}{T_s}$.

One can always sample a signal this way. However, the main question is how fast one needs to sample (i.e., how small T_s needs to be) in order for the samples to be a faithful representation of the underlying continuous-time signal. We will study this question here.

First, it will be useful for us to have a mathematical representation of the sampled signal. Define the impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s).$$

Then, sampling a signal $x(t)$ at a sampling period of T_s can be represented as multiplying that signal by an impulse train, i.e.,

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s).$$

Note that the values of the signal $x(t)$ are irrelevant outside of the points where the impulse functions in $p(t)$ occur (i.e., at the sampling instants). Let us consider the frequency spectra of these signals. Specifically, by the multiplication property of Fourier transforms, we have

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta.$$

Furthermore, since $p(t)$ is periodic, we saw that the Fourier transform of $p(t)$ will be given by

$$P(j\omega) = \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s),$$

as the Fourier series coefficients of $p(t)$ are each $\frac{1}{T_s}$. Thus,

$$\begin{aligned} X_p(j\omega) &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\theta)\delta(\omega - \theta - n\omega_s)d\theta \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(j(\omega - n\omega_s)). \end{aligned}$$

Thus, the frequency spectrum of $x_p(t)$ consists of copies of the frequency spectrum of $x(t)$, where each copy is shifted (in frequency) by an integer multiple of the sampling frequency ω_s and scaled by $\frac{1}{T_s}$ (see Fig. 7.1).

If we want to be able to reconstruct $x(t)$ from its sampled version $x_p(t)$, we would like to make sure that there is an exact copy of $X(j\omega)$ that can be extracted from $X_p(j\omega)$. Based on the above discussion, we see that this will be the case if no two copies of $X(j\omega)$ overlap in $X_p(j\omega)$. Looking at Fig. 7.1, this will occur as long as

$$\omega_s - \omega_M > \omega_M,$$

or equivalently,

$$\omega_s > 2\omega_M,$$

where ω_M is the largest frequency at which $x(t)$ has nonzero content. This leads to the **sampling theorem**.

> If the sampling frequency ω_s is larger than twice the largest frequency of the signal $x(t)$, then we can reconstruct the signal $x(t)$ from its sampled version $x_p(t)$ by passing $x_p(t)$ through an ideal low-pass filter, with cutoff $\omega_c = \frac{\omega_s}{2}$.

The frequency $2\omega_M$ is called the **Nyquist rate**.

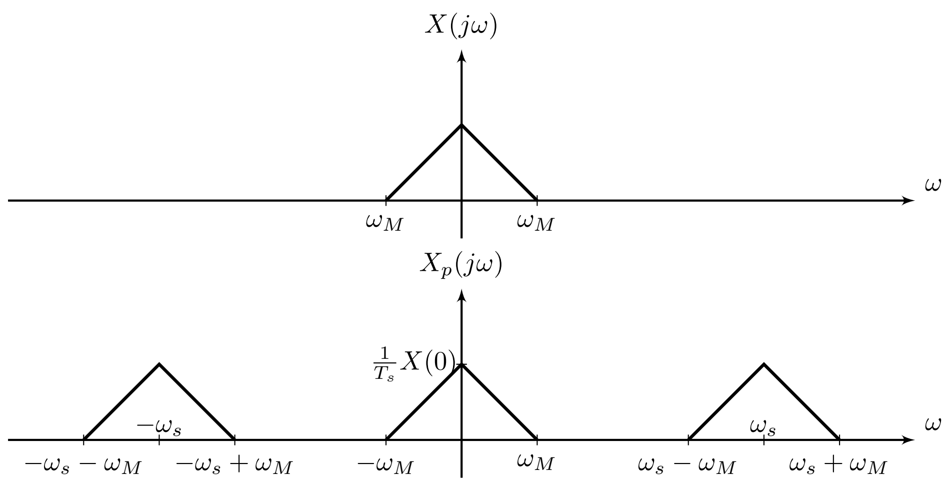


Figure 7.1: The frequency spectrum of the signal $x(t)$ and the signal $x_p(t)$.

7.2 Reconstruction of a Signal From Its Samples

In general, it is not possible to implement an ideal low-pass filter: obtaining sharp cutoffs is difficult, and furthermore, an ideal low-pass filter is noncausal (as it corresponds to a sinc function in the time-domain). There are various other options that are frequently used to reconstruct sampled signals.

7.2.1 Zero-Order Hold

The simplest option to reconstruct a signal is to simply hold the value of the signal constant at the value of the previous sample. This is called a **zero-order-hold** (ZOH). To compare this strategy to the ideal low-pass-filter, let's consider the transfer function of the ZOH. Specifically, note that if we put an impulse function into the ZOH, the output $h_0(t)$ (i.e., impulse response) will be a square pulse from $t = 0$ to $t = T_s$ (because the ZOH will keep the value of the sample at $t = 0$ constant at 1 until the next sample at $t = T_s$, after which point all samples are zero). This is shown in Fig. 7.2. It is easy to check that the transfer function is given by

$$H_0(j\omega) = \int_{-\infty}^{\infty} h_0(t)e^{-j\omega t}dt = e^{-j\omega\frac{T_s}{2}}\frac{\sin\left(\omega\frac{T_s}{2}\right)}{\frac{\omega}{2}}.$$

This has magnitude T_s at $\omega = 0$ (like the ideal reconstructor), and the first frequency at which it is equal to zero is at ω_s (unlike the ideal reconstructor that cuts off at $\frac{\omega_s}{2}$). Furthermore, this frequency spectrum is not bandlimited, and thus the copies of $X(j\omega)$ in the spectrum of $X_p(j\omega)$ will leak into the reconstructed signal under the ZOH.

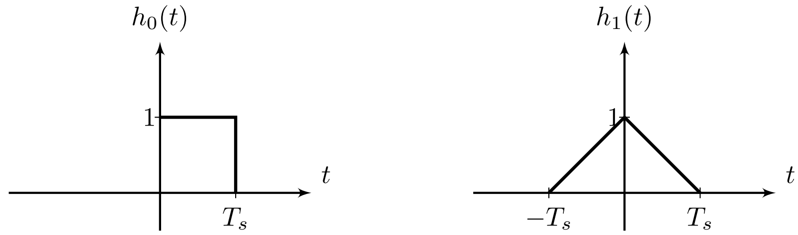


Figure 7.2: The impulse response of a zero-order-hold (left) and a first-order-hold (right).

7.2.2 First-Order Hold

A slightly more sophisticated reconstruction mechanism is to create a continuous-time signal by joining each consecutive pair of samples by a line. This is called a **first-order-hold** (FOH). The impulse response $h_1(t)$ of an FOH is shown in the right plot of Fig. 7.2 (again, imagine how the FOH reacts to an impulse coming into it: it sees a value of 0 at $t = -T_s$, a value of 1 at $t = 0$ and a value of 0 at $t = T_s$). The transfer function is given by

$$\begin{aligned} H_1(j\omega) &= \int_{-\infty}^{\infty} h_1(t) e^{-j\omega t} dt \\ &= \int_{-T_s}^{0} \left(\frac{1}{T_s} t + 1 \right) e^{-j\omega t} dt + \int_{0}^{T_s} \left(-\frac{1}{T_s} t + 1 \right) e^{-j\omega t} dt. \end{aligned}$$

After some algebra (integration by parts, etc.), we obtain

$$H_1(j\omega) = \frac{1}{T_s} \left(\frac{\sin\left(\omega \frac{T_s}{2}\right)}{\frac{\omega}{2}} \right)^2 = \frac{1}{T_s} |H_0(j\omega)|^2.$$

The magnitude of this filter is smaller than that of $H_0(j\omega)$ outside of $\frac{\omega_s}{2}$, although it is still not bandlimited. Furthermore, the FOH is noncausal, but can be made causal with a delay of T_s.

Higher order filters are also possible, and can be defined as a natural extension of zero and first order holds.

7.3 Undersampling and Aliasing

If the sampling frequency ω_s is not strictly larger than twice the largest frequency, we will not be able to perfect reconstruct the original signal. To illustrate this, it is easiest to consider sampled sinusoids.

Consider the signal $x(t) = \cos(t)$ which has frequency $\omega_0 = 1$. The sampling theorem indicates that as long as the sampling frequency ω_s is larger than

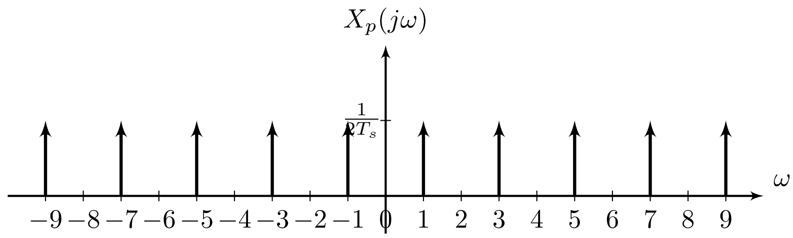


Figure 7.3: Sampling $x(t) = \cos(t)$ at a frequency of $\omega_s = 4$.

$2\omega_0 = 2$, we can reconstruct $x(t)$ from its samples. Let us choose $\omega_s = 4$. The frequency spectrum of $x_p(t) = x(t)p(t)$ is shown in Fig. 7.3.

Now consider another signal $x_1(t) = \cos(\omega_1 t)$, and suppose that we sample this signal at $\omega_s = 4$. Let $x_{p_1}(t)$ be the resulting (continuous-time) sampled signal. For what value of ω_1 will the frequency spectrum $X_{p_1}(j\omega)$ look exactly the same as $X_p(j\omega)$?

To answer this, note that $X_1(j\omega)$ has impulses located at $\pm\omega_1$, and $X_{p_1}(j\omega)$ will have impulses located at $k\omega_s \pm \omega_1$, for $k \in \mathbb{Z}$. Looking at the frequency spectrum of $X_p(j\omega)$ in Fig. 7.3, we see that ω_1 should be odd (otherwise, $X_{p_1}(j\omega)$ will have impulses at some even frequencies, whereas all of the impulses are at odd frequencies in $X_p(j\omega)$). Suppose we try $\omega_1 = 3$. Then $X_{p_1}(j\omega)$ will have impulses at ± 3, which matches two of the impulses in $X_p(j\omega)$. We should check the copies of the signals in $X_{p_1}(j\omega)$ as well. Specifically, there will be a copy centered at $\omega_s = 4$, with one impulse three units to the left (at $\omega = 1$) and one impulse three units to the right (at $\omega = 7$). Similarly, the copy centered at $2\omega_s$ will have one impulse at 5 and one impulse at 11. The same is true for the negative frequencies. Thus, we see that if $\omega_1 = 3$, then $X_{p_1}(j\omega)$ looks exactly the same as $X_p(j\omega)$, and thus the signals $x(t) = \cos(t)$ and $x_1(t) = \cos(3t)$ look exactly the same if sampled at $\omega_s = 4$.

7.4 Discrete-Time Processing of Continuous-Time Signals

Let's take a closer look at taking a signal from continuous-time, operating on it, and converting it back to discrete-time. Specifically, given a signal $x(t)$, let $x_p(t) = x(t)p(t)$ be the continuous-time representation of the sampled signal, and let $x_d[n] = x(nT_s)$ be the sequence of samples.

We have

$$X_p(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)e^{-j\omega t}dt = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s}.$$

One the other hand, if we take the discrete-time Fourier transform of the sequence $x_d[n]$, we have

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega n}.$$

Comparing the two expressions, we see that

$$X_d(e^{j\omega}) = X_p\left(j\frac{\omega}{T_s}\right).$$

In other words, the frequency spectrum of $x_p(t)$ (given by the continuous-time Fourier transform) is just a frequency-scaled version of the frequency spectrum of $x_d[n]$ (given by the discrete-time Fourier transform). Specifically, $X_p(j\omega)$ is $\omega_s = \frac{2\pi}{T_s}$ periodic, whereas $X_d(e^{j\omega})$ is 2π periodic. This scaling is essentially due to the fact that the discrete-time signal $x_d[n]$ is "normalized" with respect to the sampling period; it only operates on the sequence of samples, and does not explicitly consider how far apart those samples are. However, $x_p(t)$ explicitly contains the sampling period T_s, as the impulses are spaced that far apart.

The above result has the following implication for the digital processing of signals. Suppose that we wish to implement a filter that has a continuous-time Fourier transform $H(j\omega)$, but using a discrete-time system. Suppose $H(j\omega)$ is bandlimited, with highest frequency ω_M. Then we simply design the discrete-time filter to have frequency reponse $H_d(e^{j\omega}) = H\left(j\frac{\omega}{T_s}\right)$ for $-\omega_M T_s \leq \omega \leq \omega_M T_s$ (and $H_d(e^{j\omega})$ being 2π-periodic otherwise). The inverse Fourier transform of $H_d(e^{j\omega})$ can then be found to obtain the impulse response of the digitial filter. After the sampled signal is processed with this digital filter, it can then be transformed back into continuous-time via a ZOH, FOH, etc.

Example 7.1. Consider a bandlimited differentiator

$$H(j\omega) = \begin{cases} j\omega & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}.$$

The magnitude and phase are shown in Fig. 7.4.

To implement this in discrete-time, we create a discrete-time filter with transfer function $H_d(e^{j\omega})$ to have the same shape as $H(j\omega)$ (except for the fact that H_d is periodic), with frequencies scaled by T_s, i.e.,

$$H_d(e^{j\omega}) = H\left(j\frac{\omega}{T_s}\right) = \begin{cases} j\frac{\omega}{T_s} & |\omega| \leq \omega_c T_s \\ 0 & \omega_c T_s < |\omega| \leq 2\pi \end{cases}.$$

The impulse response of the corresponding filter is

$$h_d[n] = \begin{cases} \frac{(-1)^n}{nT_s} & n \neq 0 \\ 0 & n = 0 \end{cases}.$$

□

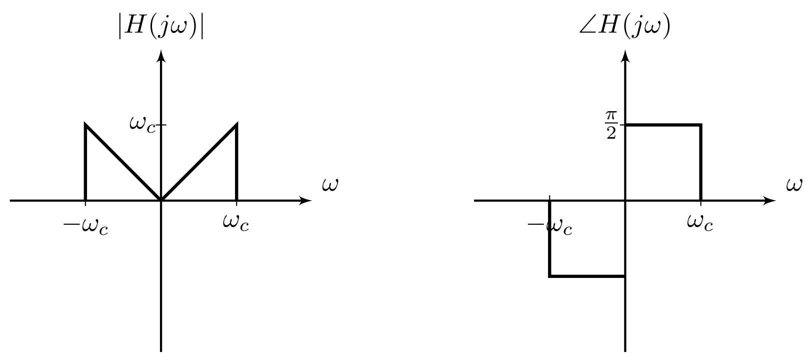


Figure 7.4: The frequency response of a bandlimited differentiator.

Chapter 8

The Laplace Transform

Reading: *Signals and Systems*, Chapter 9.1-9.3, 9.5-9.7.

Thus far, we have seen ways to take time-domain signals and transform them into frequency-domain signals, by identifying the amount of contribution of complex exponentials of given frequencies to the signal. Specifically, for periodic signals, we started with the Fourier series representation of a signal in terms of its harmonic family. For more general absolutely integrable signals, we generalized the Fourier series to the Fourier transform, where the signal is represented in terms of complex exponentials of all frequencies (not just those from the harmonic family).

8.1 The Laplace Transform

To develop this, first recall that complex exponentials of the form e^{st} are eigenfunctions of LTI systems, even when s is a general complex number. Specifically, if $x(t) = e^{st}$ is the input to an LTI system with impulse response $h(t)$, we have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(t)d\tau = e^{st}\int_{-\infty}^{\infty} h(t)e^{-s\tau}d\tau.$$

Based on the above, we see that the output is the input signal e^{st}, multiplied by the quantity $\int_{-\infty}^{\infty} h(t)e^{-s\tau}d\tau$. We will call this the Laplace transform of the signal $h(t)$.

The **Laplace transform** of a signal $x(t)$ is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt,$$

where $s \in \mathbb{C}$. We will also denote the Laplace transform of $x(t)$ by $\mathcal{L}\{x(t)\}$.

Note that the limits of the integration go from $-\infty$ to ∞, and thus this is called the **bilateral Laplace transform**. When the limits only go from 0 to ∞, it is called the **unilateral Laplace transform**. For the purposes of this course, if we leave out the qualifier, we mean the bilateral transform. Note that when $s = j\omega$, then $X(s)$ is just the Fourier transform of $x(t)$ (assuming the transform exists). However, the benefit of the Laplace transform is that it also applies to signals that do not have a Fourier transform. Specifically, note that s can be written as $s = \sigma + j\omega$, where σ and ω are real numbers. Then we have

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt.$$

Thus, for a given $s = \sigma + j\omega$, we can think of the Laplace transform as the Fourier transform of the signal $x(t)e^{-\sigma t}$. Even if $x(t)$ is not absolutely integrable, it may be possible that $x(t)e^{-\sigma t}$ is absolutely integrable if σ is large enough (i.e., the complex exponential can be chosen to cancel out the growth of the signal in the Laplace transform).

Example 8.1. Consider the signal $x(t) = e^{-at}u(t)$ where a is some real number. The Laplace transform is given by

$$\begin{aligned} X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt &= \int_{0}^{\infty} e^{-(s+a)t}dt \\ &= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty \\ &= \frac{1}{s+a}, \end{aligned}$$

as long as $\text{Re}\{s+a\} > 0$, or equivalently $\text{Re}\{s\} > -a$. Note that if a is positive, then the integral converges for $\text{Re}\{s\} = 0$ as well, in which case we get the Fourier transform $X(j\omega) = \frac{1}{j\omega+a}$. However, if a is negative, then the signal does not have a Fourier transform (but it does have a Laplace transform for s with a sufficiently large real part). □

Example 8.2. Consider the signal $x(t) = -e^{-at}u(-t)$ where a is a real number.

We have

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$
$$= \frac{1}{s+a}\, e^{-(s+a)t}\Big|_{-\infty}^{0}$$
$$= \frac{1}{s+a},$$

as long as $\mathrm{Re}\{s+a\} < 0$, or equivalently, $\mathrm{Re}\{s\} < -a$. □

Comparing the above examples, we notice that both the signals $e^{-at}u(t)$ and $-e^{-at}u(-t)$ had the same Laplace transform $\frac{1}{s+a}$, but that the ranges of s for which each had a Laplace transform was different.

> Consider a signal $x(t)$. The range of values of s for which the Laplace transform integral converges is called the **Region of Convergence** (ROC) of the Laplace transform.

Thus, in order to specify the Laplace transform of a signal, we have to specify both the algebraic expression (e.g., $\frac{1}{s+a}$) and the region of convergence for which this expression is valid. A convenient way to visualize the ROC is as a shaded region in the complex plane. For example, the ROC $\mathrm{Re}\{s\} > -a$ can be represented by shading all of the complex plane to the right of the line $\mathrm{Re}\{s\} = -a$. Similarly, the ROC $\mathrm{Re}\{s\} < -a$ is represented by shading the complex plane to the left of the line $\mathrm{Re}\{s\} = -a$.

Example 8.3. Consider the signal $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$. It is easy to see that the Laplace transform is a linear operation, and thus we can find the Laplace transform of $x(t)$ as a sum of the Laplace transform of the two signals on the right hand side.

The Laplace transform of $3e^{-2t}u(t)$ is $\frac{3}{s+2}$, with ROC $\mathrm{Re}\{s\} > -2$. The Laplace transform of $-2e^{-t}u(t)$ is $-\frac{2}{s+1}$, with ROC $\mathrm{Re}\{s\} > -1$. Thus, for the Laplace transform of $x(t)$ to exist, we need s to fall in the ROC of both of its constituent parts, which means $\mathrm{Re}\{s\} > -1$. Thus,

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2},$$

with ROC $\mathrm{Re}\{s\} > -1$. □

In the above examples, we saw that the Laplace transform was of the form

$$X(s) = \frac{N(s)}{D(s)},$$

where $N(s)$ and $D(s)$ are polynomials in s. The roots of the polynomial $N(s)$ are called the **zeros** of $X(s)$ (since $X(s)$ will be zero when s is equal to one of those roots), and the roots of $D(s)$ are called the **poles** of $X(s)$ (evaluating $X(s)$ at a pole will yield ∞). We can draw the poles and zeros in the s-plane using $\circ$ for zeros and $\times$ for poles.

Example 8.4. Consider the signal

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t).$$

The Laplace transform of $\delta(t)$ is

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

for any value of s. Thus the ROC for $\delta(t)$ is the entire s-plane. Putting this with the other two terms, we have

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} = \frac{(s-1)^2}{(s+1)(s-2)},$$

with ROC $\text{Re}\{s\} > 2$. □

8.2 The Region of Convergence

Let us dig a little deeper into the region of convergence for Laplace transforms. Recall that for a given signal $x(t)$, the ROC is the set of values $s \in \mathbb{C}$ such that the Laplace transform integral converges. More specifically, writing $s = \sigma + j\omega$, we see that

$$\int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(x(t)e^{-\sigma t}\right) e^{-j\omega t}dt.$$

Thus, as long as $x(t)e^{-\sigma t}$ is absolutely integrable, this integral exists. Note that this does not depend on the value of ω. Thus, we have the following fact about the ROC.

Property 1. The ROC consists of strips parallel to the $j\omega$-axis in the s-plane.

For the next property of the ROC, suppose that the signal $x(t)$ has a Laplace transform given by a rational function. We know that the poles of this function are the set of complex s such that $X(s)$ is infinite. Since $X(s)$ is given by the Laplace transform integral, we see that the ROC cannot contain any poles of $X(s)$.

Property 2. For rational Laplace transforms, the ROC does not contain any poles of $X(s)$.

The third property pertains to signals that are of finite duration (and absolutely integrable). Specifically, suppose that $x(t)$ is nonzero only between two finite times T_1 and T_2. Then we have

$$X(s) = \int_{T_1}^{T_2} x(t)e^{-st}dt$$

which is finite for any finite s. Thus we have the following.

> **Property 3.** If $x(t)$ is of finite duration and absolutely integrable, then the ROC is the entire s-plane.

Another way to think of the above property is as follows. No matter what σ we pick, the signal $x(t)e^{-\sigma t}$ will be absolutely integrable as long as $x(t)$ is of finite duration and absolutely integrable. The fact that $x(t)$ is of finite duration allows us to overcome the fact that the signal $e^{-\sigma t}$ may be growing unboundedly outside of the interval $[T_1, T_2]$.

While the previous property considered the case where the signal is of finite duration, we will also be interested in signals that are only zero either before or after some time. First, a signal $x(t)$ is *right-sided* if there exists some $T_1 \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T_1$. A signal $x(t)$ is *left-sided* if there exists some $T_2 \in \mathbb{R}$ such that $x(t) = 0$ for all $t > T_2$. A signal $x(t)$ is *two-sided* if it extends infinitely far in both directions.

> **Property 4.** If $x(t)$ is right-sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC contains all values s such that $\mathrm{Re}\{s\} \geq \sigma_0$.

To see why this is true, first note that since $x(t)$ is right-sided, there exists some T_1 such that $x(t) = 0$ for all $t < T_1$. If s with $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then $x(t)e^{-\sigma_0 t}$ is absolutely integrable, i.e.,

$$\int_{T_1}^{\infty} |x(t)|e^{-\sigma_0 t}dt < \infty.$$

Now suppose that we consider some $\sigma_1 > \sigma_0$. If $T_1 > 0$, then $e^{-\sigma_1 t}$ is always smaller than $e^{-\sigma_0 t}$ over the region of integration, and thus $x(t)e^{-\sigma_1 t}$ will also be absolutely integrable. If $T_1 < 0$, then

$$\begin{aligned}\int_{T_1}^{\infty} |x(t)|e^{-\sigma_1 t}dt &= \int_{T_1}^{0} |x(t)|e^{-\sigma_1 t}dt + \int_{0}^{\infty} |x(t)|e^{-\sigma_1 t}dt \\ &\leq \int_{T_1}^{0} |x(t)|e^{-\sigma_1 t}dt + \int_{0}^{\infty} |x(t)|e^{-\sigma_0 t}dt.\end{aligned}$$

The first term is finite (since it is integrating some signal of finite duration), and the second term is finite since $x(t)e^{-\sigma_0 t}$ is absolutely integrable. Thus, once again, $x(t)e^{-\sigma_1 t}$ is absolutely integrable, and thus s with $\text{Re}\{s\} \geq \sigma_0$ also falls within the ROC of the signal.

The same reasoning applies to show the following property.

> **Property 5.** If $x(t)$ is left-sided and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC contains all values s such that $\text{Re}\{s\} \leq \sigma_0$.

If $x(t)$ is two-sided, we can write $x(t)$ as $x(t) = x_R(t) + x_L(t)$, where $x_R(t)$ is a right-sided signal and $x_L(t)$ is a left-sided signal. The former has an ROC that is the region to the right of some line in the s-plane, and the latter has an ROC that is the region to the left of some line in the s-plane. Thus, the ROC for $x(t)$ contains the intersection of these two regions (if there is no intersection, $x(t)$ does not have a Laplace transform).

> **Property 6.** If $x(t)$ is two-sided and contains the line $\text{Re}\{s\} = \sigma_0$ in its ROC, then the ROC consists of a strip in the s-plane that contains the line $\text{Re}\{s\} = \sigma_0$.

Example 8.5. Consider the signal $x(t) = e^{-b|t|}$. We write this as

$$x(t) = e^{-bt}u(t) + e^{bt}u(-t).$$

Note that we modify the definition of $u(t)$ in this expression so that $u(0) = \frac{1}{2}$, so that $x(0) = 1$ as required. As this modification is only at a single point (of zero width and finite height), it will not make a difference to the quantities calculated by integrating the signals.

The signal $e^{-bt}u(t)$ has Laplace transform

$$\mathcal{L}\{e^{-bt}u(t)\} = \frac{1}{s+b},$$

with ROC $\text{Re}\{s\} > -b$. The signal $e^{bt}u(-t)$ has Laplace transform

$$\mathcal{L}\{e^{bt}u(-t)\} = \frac{-1}{s-b},$$

with ROC $\text{Re}\{s\} < b$. If $b \leq 0$, then these two ROCs do not overlap, in which case $x(t)$ does not have a Laplace transform. However, if $b > 0$, then $x(t)$ has the Laplace transform

$$\mathcal{L}\{x(t)\} = \frac{1}{s+b} - \frac{1}{s-b},$$

with ROC $-b < \text{Re}\{s\} < b$. □

As we will see soon, a rational Laplace transform $X(s)$ can be decomposed into a sum of terms, each of which correspond to an exponential signal. The ROC for $X(s)$ consists of the intersection of the ROCs for each of those terms, and since none of the ROCs can contain poles, we have the following property.

> **Property 7.** If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinty.

Example 8.6. Consider

$$X(s) = \frac{1}{s(s+1)}.$$

There are three possible ROCs for this Laplace transform: the region to the right of the line $\text{Re}\{s\} = 0$, the region between the lines $\text{Re}\{s\} = -1$ and $\text{Re}\{s\} = 0$, or the region to the left of the line $\text{Re}\{s\} = -1$. □

8.3 The Inverse Laplace Transform

Consider again the Laplace transform evaluated at $s = \sigma + j\omega$:

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt.$$

Since this is just the Fourier transform of $x(t)e^{-\sigma t}$, we can use the inverse Fourier transform formula to obtain

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega.$$

If we multiply both sides by $e^{\sigma t}$, we get

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega.$$

Doing a change of variable $s = \sigma + j\omega$, we get

$$x(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + \infty} X(s)e^{st}d\omega.$$

This is the **inverse Fourier transform formula**. It involves an integration over the line in the complex plane consisting of points satisfying $\text{Re}\{s\} = \sigma$. There are actually simpler ways to calculate the inverse Fourier transform, using the notion of partial fraction expansion, which we will consider here.

Example 8.7. Consider $X(s) = \frac{1}{s(s+1)}$. First, we note that

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Now each of these terms is of a form that we know (they correspond to complex exponentials). So, for example, if the ROC for $X(s)$ is the region to the right of the imaginary axis, since the ROC consists of the intersection of the ROCs of both of the terms, we know that both terms must be right-sided signals. Thus,

$$x(t) = u(t) - e^{-t}u(t).$$

Similarly, if the ROC is between the lines $\text{Re}\{s\} = -1$ and $\text{Re}\{s\} = 0$, the first term is left-sided and the second term is right-sided, which means

$$x(t) = -u(-t) - e^{-t}u(t).$$

Finally, if the ROC is to the right of the line $\text{Re}\{s\} = -1$, both terms are left-sided and thus

$$x(t) = -u(-t) + e^{-t}u(-t).$$

□

In the above example, we "broke up" the function $\frac{1}{s(s+1)}$ into a sum of simpler functions, and then applied the inverse Laplace Transform to each of them. This is a general technique for inverting Laplace Transforms, which we now study.

8.3.1 Partial Fraction Expansion

Suppose we have a *rational* function

$$X(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} \ ,$$

where the a_i's and b_i's are constant real numbers.

Definition 8.1. If $m \leq n$, the rational function is called *proper*. If $m < n$, it is *strictly proper*. □

By factoring $N(s)$ and $D(s)$, we can write

$$X(s) = K\frac{(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} \ .$$

Recall that the zeros of $X(s)$ are given by $-z_1, -z_2, \ldots, -z_m$, and the poles are $-p_1, -p_2, \ldots, -p_n$. First, suppose each of the poles are distinct and that $X(s)$ is strictly proper. We would like to write

$$X(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n} \ ,$$

for some constants $k_1, k_2, \ldots, k_n$, since the inverse Laplace Transform of $X(s)$ is easy in this form. How do we find $k_1, k_2, \ldots, k_n$?

Heaviside's Cover-up Method. To find the constant k_i, multiply both sides of the expansion of $X(s)$ by $(s + p_i)$:

$$(s+p_i)X(s) = \frac{k_1(s+p_i)}{s+p_1} + \frac{k_2(s+p_i)}{s+p_2} + \cdots + k_i + \cdots + \frac{k_n(s+p_i)}{s+p_n} \quad .$$

Now if we let $s = -p_i$, then all terms on the right hand side will be equal to zero, except for the term k_i. Thus, we obtain

$$k_i = (s+p_i)X(s)|_{s=-p_i} \quad .$$

Example 8.8. Consider $X(s) = \frac{s+5}{s^3+3s^2-6s-8}$. The denominator is factored as

$$s^3 + 3s^2 - 6s - 8 = (s+1)(s-2)(s+4).$$

We would thus like to write

$$X(s) = \frac{s+5}{(s+1)(s-2)(s+4)} = \frac{k_1}{s+1} + \frac{k_2}{s-2} + \frac{k_3}{s+4}.$$

Using the Heaviside coverup rule, we obtain

$$k_1 = (s+1)X(s)|_{s=-1} = \frac{4}{(-3)(3)} = -\frac{4}{9}$$
$$k_2 = (s-2)X(s)|_{s=2} = \frac{7}{(3)(6)} = \frac{7}{18}$$
$$k_3 = (s+4)X(s)|_{s=-4} = \frac{1}{(-3)(-6)} = \frac{1}{18}.$$

□

The partial fraction expansion when some of the poles are repeated is obtained by following a similar procedure, but it is a little more complicated. We will not worry too much about this scenario here. One can also do a partial fraction expansion of nonstrictly proper functions by first dividing the denominator into the numerator to obtain a constant and a strictly proper function, and then applying the above partial fraction expansion.

8.4 Some Properties of the Laplace Transform

The Laplace transform has various properties that are quite similar to those for Fourier transforms (linearity, time-shifting, etc.) We will focus on two important ones here.

8.4.1 Convolution

Consider two signals $x_1(t)$ and $x_2(t)$ with Laplace transforms $X_1(s)$ and $X_2(s)$ and ROCs R_1 and R_2, respectively. Then

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)X_2(s),$$

with ROC containing $R_1 \cap R_2$. Thus, convolution in the time-domain maps to multiplication in the s-domain (as was the case with Fourier transforms).

Example 8.9. Consider the convolution $u(t) * u(t)$. Since $\mathcal{L}\{u(t)\} = \frac{1}{s}$ with ROC $\text{Re}\{s\} > 0$, we have

$$\mathcal{L}\{u(t) * u(t)\} = \frac{1}{s^2},$$

with ROC containing the region $\text{Re}\{s\} > 0$. □

8.4.2 Differentiation

Consider a signal $x(t)$, with Laplace transform $X(s)$ and ROC R. We have

$$x(t) = \frac{1}{2\pi}\int X(s)e^{st}ds.$$

Differentiating both sides with respect to t, we have

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int sX(s)e^{st}ds.$$

Thus, we see that

$$\mathcal{L}\{\frac{dx}{dt}\} = sX(s),$$

with ROC containing R. More generally,

$$\mathcal{L}\{\frac{d^m x}{dt^m}\} = s^m X(s) \enspace .$$

8.4.3 Integration

Given a signal $x(t)$ whose Laplace transform has ROC R, consider the integral $\int_{-\infty}^{t} x(\tau)d\tau$. Note that

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t),$$

and thus using the convolution property, we have

$$\mathcal{L}\{\int_{-\infty}^{t} x(\tau)d\tau\} = \frac{1}{s}X(s),$$

with ROC containing $R \cap \{\text{Re}\{s\} > 0\}$.

8.5 Finding the Output of an LTI System via Laplace Transforms

The Laplace transform properties (time-domain convolution corresponding to frequency-domain multiplication, in particular) are very useful in analyzing the output of LTI systems to inputs. Specifically, consider an LTI system with impulse response $h(t)$ and input $y(t)$. We know that

$$Y(s) = H(s)X(s),$$

assuming all Laplace transforms exist. Using the expressions for $H(s)$ and $X(s)$, we can thus calculate $Y(s)$ (and its ROC), and then use an inverse Laplace transform to determine $y(t)$.

Example 8.10. Consider an LTI system with impulse response $h(t) = e^{-2t}u(t)$. Suppose the input is $x(t) = e^{-3t}u(t)$. The Laplace transforms of $h(t)$ and $x(t)$ are

$$H(s) = \frac{1}{s+2}, \quad X(s) = \frac{1}{s+3},$$

with ROCs $\text{Re}\{s\} > -2$ and $\text{Re}\{s\} > -3$, respectively. Thus we have

$$Y(s) = H(s)X(s) = \frac{1}{s+2}\frac{1}{s+3},$$

with ROC $\text{Re}\{s\} > -2$. Using partial fraction expansion, we have

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+3},$$

and thus $y(t) = e^{-2t}u(t) - e^{-3t}u(t)$. □

Example 8.11. Consider an LTI system with impulse response $h(t) = -e^{4t}u(-t)$ and input $x(t) = e^{2t}u(t)$, where we interpret $u(-t)$ as being 1 for $t < 0$. The Laplace transforms are

$$H(s) = \frac{1}{s-4}, \quad X(s) = \frac{1}{s-2},$$

with ROCs $\text{Re}\{s\} < 4$ and $\text{Re}\{s\} > 2$, respectively. Since there is a nonempty intersection, we have

$$Y(s) = H(s)X(s) = \frac{1}{s-4}\frac{1}{s-2} = \frac{1}{2}\frac{1}{s-4} - \frac{1}{2}\frac{1}{s-2},$$

with ROC $2 < \text{Re}\{s\} < 4$. Thus, $y(t)$ is two-sided, and given by

$$y(t) = -\frac{1}{2}e^{4t}u(-t) + \frac{1}{2}e^{2t}u(t).$$

□

8.6 Finding the Impulse Response of a Differential Equation via Laplace Transforms

The differentiation property of Laplace transforms is also extremely useful for analyzing differential equations. Specifically, suppose that we have a constant-coefficient differential equation of the form

$$\sum_{k=0}^{n} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k x(t)}{dt^k}$$

Taking Laplace transforms, we obtain

$$(\sum_{k=0}^{n} a_k s^k) Y(s) = (\sum_{k=0}^{m} b_k s^k) X(s)$$

or equivalently

$$Y(s) = \underbrace{\frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}}_{H(s)} X(s).$$

Thus, the impulse response of the differential equation is just the inverse Laplace transform of $H(s)$ (corresponding to an appropriate region of convergence).

Example 8.12. Consider the differential equation

$$\frac{d^3 y(t)}{dt^3} + 2\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

Taking Laplace transforms, we have

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^3 + 2s^2 - s - 2} = \frac{1}{(s-1)(s+1)(s+2)}.$$

In this case, we have the partial fraction expansion

$$H(s) = \frac{1}{6}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+2}$$

Suppose we are told the impulse response is causal (which implies it is right-sided). Thus, the ROC would be to the right of the furthest pole and we have

$$h(t) = \left(\frac{1}{6}e^{t} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-2t}\right) u(t).$$

□

Bibliography