Realization of Digital Filters
Discrete Time Systems: Overview

• The difference equation, the impulse response and the system function are equivalent characterization of the input/output relation of a LTI Discrete-time systems.

• LTI system can be modeled using:
  1. A Difference/Differential equation, \( y(n) = x[n] + x[n-1] + \ldots \)
  2. Impulse Response, \( h(n) \)
  3. Transfer Function, \( H(z) \)

• The systems that are described by the difference equations, can be represented by structures consisting of an interconnection of the basic operations of addition, multiplication by a constant or signal multiplication, delay and advance.
• The Adder, Multiplier, Delay & Advance is shown below:

1. Adder:

\[ w[n] = x[n] + y[n] \]

2. Multiplier:

\[ w[n] = x[n] y[n] \]

Modulator:
3. Delay :

$$x[n] \xrightarrow{Z^{-1}} x[n-1]$$

4. Advance :

$$x[n] \xrightarrow{Z} x[n+1]$$
• Consider a first-order causal LTI IIR digital filter described by

\[ y[n] = -d_1 y[n-1] + p_0 x[n] + p_1 x[n-1] \]

The block diagram representation is
Example 1:
Given an LTI system is described by the difference equation below:

\[ y(n) - 0.25y(n-1) = 0.5x(n) + 0.5x(n-1) \]

Draw the block diagram representation
The implementation of LTI system can be realized in terms of **Block Diagram** and **Signal Flow Graph**.

The LTI system can be represented in 2 manner:

a. Block Diagram

b. Mathematical Model
• Example of Block diagram is shown below:
Block Diagrams

- Useful way to illustrate implementations
- Z-transform helps analysis:
  \[ Y(z) = G_1(z)[X(z) + G_2(z)Y(z)] \]
  \[ \Rightarrow Y(z)[1 - G_1(z)G_2(z)] = G_1(z)X(z) \]
  \[ \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{G_1(z)}{1 - G_1(z)G_2(z)} \]

- Approach
  - Output of summers as dummy variables
  - Everything else is just multiplicative
Equivalent Structures

- Modifications to block diagrams that do not change the filter
  - e.g. Commutation \( H = AB = BA \)

- Factoring \( AB + CB = (A+C)B \)

fewer blocks

less computation
The Transfer Function of LTI system can be connected in 2 ways:

a. Parallel Connection:

\[
H(z) = H_1(z) + H_2(z) + \ldots + H_L(z)
\]

The overall transfer function,

\[
H(z) = H_1(z) + H_2(z) + \ldots + H_L(z)
\]
b. Cascade connection:

The overall transfer function:
\[ H(z) = H_1(z) \cdot H_2(z) \cdot \ldots \cdot H_L(z) \]

Each one of them can be implemented using any of the Direct Forms.
• Canonic
  – number of delays in the block diagram representation is equal to the order of the difference equation

• Non-canonic
  – otherwise
FIR FILTER STRUCTURES
FIR FILTERS

• These are realized using only two Forms:
• (as it only has the Numerator part i.e. ALL ZERO SYSTEMS)

• 1. Direct Form 1 or Tapped delay Line or Transversal delay Line Filter.

• 2. Cascade form
FIR Filter Structures

- Direct form
  - An FIR filter of order N requires N + 1 multipliers, N adders and N delays.
  - An FIR filter of order 4

FIR Filter Structures

- Direct form “Tapped Delay Line”

\[
y[n] = h_0 x[n] + h_1 x[n-1] + \ldots = \sum_{k=0}^{4} h_k x[n-k]
\]

- Transpose

- Re-use delay line if several inputs \(x_i\) for single output \(y\)
• Cascade form
  – Transfer function $H(z)$ of a causal FIR filter of order $N$

  \[ H(z) = \sum_{k=0}^{N} h[k] z^{-k} \]

  – Factorized form

  \[ H(z) = h[0] \prod_{k=1}^{k} (1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}) \]

  Where $k = N/2$ if $N$ is even and $k = (N + 1)/2$ if $N$ is odd, with $\beta_{2k} = 0$
Example…

• Determine the Direct Form & Cascade Form Realization for the transfer Function of an FIR Digital filter which is given by

\[ H(z) = (1 - \frac{1}{4} Z + \frac{3}{8} Z^2)(1 - \frac{1}{8}Z - \frac{1}{2}Z^2) \]
Direct Form

- We Simply Expand the equation to get this form as

\[
H(z) = 1 - \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} + \frac{5}{64}z^{-3} - \frac{3}{16}z^{-4}
\]

This function can be realised in FIR direct form as depicted in figure 8.72.
Cascade Form

- $H(z) = H_1(z) * H_2(z)$ & hence
Equivalent Structures

- Transpose
  - reverse paths
  - adders↔nodes
  - input↔output

\[
Y = b_1 X + b_2 z^{-1} X + b_3 z^{-2} X \\
= b_1 X + z^{-1} \left( b_2 X + z^{-1} b_3 X \right)
\]
IIR FILTER STRUCTURES
IIR system/filter can be realized in several structures:

1. DIRECT FORM I
2. DIRECT FORM II (CANONIC)
3. CASCADE FORM
4. PARALLEL FORM
IIR System Function

\[ H(z) = \frac{Y(z)}{S(z)} = \frac{\sum_{k=0}^{M-1} B_k z^{-k}}{1 + \sum_{k=1}^{N-1} A_k z^{-k}} \]

where \( M \) and \( N \) are integer numbers.
Bifurcation of $H(z)$ into $H_1(z)$ & $H_2(z)$

$$H(z) = \frac{\sum_{k=0}^{M-1} B_k z^{-k}}{1 + \sum_{k=1}^{N-1} A_k z^{-k}} = H_1(z) \cdot H_2(z) \quad \text{...(7.3)}$$

$$H_1(z) = \sum_{k=0}^{M-1} B_k z^{-k} \quad \text{...(7.4)}$$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N-1} A_k z^{-k}}$$

$$= \left[1 + \sum_{k=1}^{N} A_k z^{-k}\right]^{-1} = 1 - \sum_{k=1}^{N} A_k z^{-k} = 1 + \sum_{k=1}^{N} (-A_k) z^{-k} \quad \text{...(7.5)}$$
Block Diagram of Direct Form I & II
IIR Filter Structures

- IIR: numerator + denominator

\[ H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \ldots}{1 + d_1 z^{-1} + d_2 z^{-2} + \ldots} \]

\[ = P(z) \cdot \frac{1}{D(z)} \]

FIR

all-pole IIR
IIR Filter Structures

- Hence, Direct form I

- Commutation → Direct form II (DF2)
  - same signal
  - delay lines merge
  - “canonical”
  - min. memory usage
Direct Form I Realization

(a) Diagram showing the structure of an all-zero system $H_1(z)$ followed by an all-pole system $H_2(z)$.

(b) Detailed block diagram of the all-zero system $H_1(z)$ and the all-pole system $H_2(z)$, including coefficients $B_0, B_1, B_2, B_3, B_{M-1}, B_M, A_1, A_2, A_3, A_{M-1}, A_N$.
Direct Form II Realization
Canonic Direct Form II Realization
Parallel Form Realization

\[ H_k(z) = \frac{B_{k0} + B_{k1}z^{-1}}{1 + A_{k1}z^{-1} + A_{k2}z^{-2}} \]  ...(7.9)

Coefficients \( B_{ki} \) and \( A_{ki} \) real-valued system parameters.

Parallel form network structures are shown in Fig. 7.9 and Fig. 7.10.

Fig. 7.9 Parallel-form network structure of IIR system.

Fig. 7.10 Structure of II\(\text{nd} \) order section in a parallel-form network structure realization.
Example 7.2 Sketch the direct form-I, direct form-II, cascade and parallel-form network structures for the system characterized by the following difference equations.

\[ y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + s(n) + \frac{1}{3} s(n-1) \]
(a) Block diagram of direct form-I of above problem.
(b) Direct form-I, network structure of above filter.
Direct Form II & Cascade Form

Fig. 7.12 Direct form-II realization of above filter.

Fig. 7.13 Cascade-form network structure of above filter.
Parallel Form

• To get this we use PFE method:

• \( H(z) = \frac{Y(z)}{X(z)} \)

• Giving us \( A1 = -7/3 \) & \( A2 = 10/3 \) so implementing it we have ----
Parallel Form

Fig. 7.14 Parallel form network structure of above filter.
• **Direct Form I**
  – Consider a third order IIR described by transfer function

\[
H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_3 z^{-3}}
\]

  – Implement as a cascade of two filter section
Where

\[ H_1(z) = \frac{W(z)}{X(z)} = P(z) = p_0 + p_1z^{-1} + p_2z^{-2} + p_3z^{-3} \]

and

\[ H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{D(z)} = \frac{1}{1 + d_1z^{-1} + d_2z^{-2} + d_3z^{-3}} \]

- Resulting in realization indicated below
• Direct Form I
• Direct Form II (Canonic)

– The two top delays can be shared
• Cascade Form

\[ H(z) = p_0 \prod_k \left( \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right) \]

• A third order transfer function

\[ H(z) = p_0 \left( \frac{1 + \beta_{11} z^{-1}}{1 + \alpha_{11} z^{-1}} \right) \left( \frac{1 + \beta_{12} z^{-1} + \beta_{22} z^{-2}}{1 + \alpha_{12} z^{-1} + \alpha_{22} z^{-1}} \right) \]
• Cascade Form
• Parallel Form: Use Partial Fraction Expansion Form to realize them

\[ H(z) = \gamma_0 + \sum_k \left( \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right) \]
- Parallel Form: used in High Speed Filtering applications (as operated parallely)
Parallel IIR Structures

- Can express $H(z)$ as sum of terms (IZT)

$$H(z) = \text{consts} + \sum_{\ell=1}^{N} \frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \quad \rho_\ell = (1 - \lambda_\ell z^{-1})F(z)|_{z=\lambda_\ell}$$

- Or, second-order terms:

$$H(z) = \gamma_0 + \sum_k \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

- Suggests parallel realization...
Parallel IIR Structures

- Sum terms become parallel paths
- Poles of each SOS are from full TF
- System zeros arise from output sum
- Why do this?
  - stability/sensitivity
  - reuse common terms